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Hamilton cycles in random lifts of graphs

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ABSTRACT

For a graph G the random n-lift of G is obtained by replacing each of its vertices by a set of n vertices, and joining a pair of sets by a random matching whenever the corresponding vertices of G are adjacent. We show that asymptotically almost surely the random lift of a graph G is Hamiltonian, provided G has the minimum degree at least 5 and contains two disjoint Hamiltonian cycles whose union is not a bipartite graph.

1. Introduction

The notion of a random lift was proposed by Amit and Linial [1] as a discrete version of the topological notion of covering maps, which are "locally bijective" homomorphisms. For graphs G and H, a map $\pi : V(H) \to V(G)$ is a *covering map* from H to G if for every $v \in V(H)$ the restriction of π to the neighborhood of v is a bijection onto the neighborhood of $\pi(v) \in V(G)$. In particular, for every vertex $v \in V(H)$ the degree of v must be the same as the degree of $\pi(v)$. The set of all vertices which are mapped onto a vertex v is called the *fiber above* v and denoted by $\tilde{G}_v$. Since the term covering has been already widely used in graph theory, following Amit and Linial, we use the term *lift* instead. For instance, we say that H from the previous example is a lift of G. We often denote the lift of G by $\tilde{G}$.

If for every vertex $v \in G$ the fiber $\tilde{G}_v$ has size n, then we call such a lift an *n-lift*. We denote the set of all n-lifts of a given graph G by $L_n(G)$ and call G the *base graph*. A random n-lift of G is a graph chosen uniformly at random from the set $L_n(G)$. This is equivalent to associating with each vertex $u \in G$ a set $\tilde{G}_u$ of n vertices and independently connecting each pair $(\tilde{G}_u, \tilde{G}_v)$ by a random matching whenever u and v are adjacent in the base graph G. Another way to describe this process is to take $\tilde{G}_v = \{v_1, \dots, v_n\}$ and $\tilde{G}_u = \{u_1, \dots, u_n\}$, choose uniformly at random one of the $n!$ permutations

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$\sigma_{vu} : [n] \to [n]$, and connect v_i with $u_{\sigma_{vu}(i)}$. Note that such permutations (or matchings) are chosen independently for each edge uv in G.

Our interest lies in the asymptotic properties of lifts of graphs, as parameter n goes to infinity. In particular, we say that a property holds *asymptotically almost surely*, or, briefly, *aas*, if its probability tends to 1 as n tends to infinity. Sometimes, instead of saying that the random lift of G has aas a property $\mathcal{A}$, we write that almost every random lift of a graph G has $\mathcal{A}$.

The first paper in the theory of random lifts of graphs dealt with their connectivity properties. Amit and Linial [1] have proven that if G is a simple, connected graph with minimum degree $\delta \geq 3$, then its random lift is aas δ-connected. It was shown in [14] that for a graph G with $\delta(G) \geq 2k - 1$ we have an even stronger property: namely a random lift of G is aas k-linked. The term *k-linked* refers to a graph with the property that for every $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ the graph contains k vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i connects s_i to t_i, $1 \leq i \leq k$.

Only a few other properties of random lifts have been studied, such as expansion properties [2], matchings [12], and the independence and chromatic numbers [3]. Here we consider the property that a graph contains a Hamilton cycle. The two main problems concerning Hamiltonicity of random lifts have been stated by Linial [11], who asked the following two questions:

Problem 1. Is it true that for a given G the random lift $L_n(G)$ is either aas Hamiltonian, or aas non Hamiltonian?

Problem 2. Let G be a connected d-regular graph with $d \geq 3$. Is it true that random n-lift of G is aas Hamiltonian?

Burgin, Chebolu, Cooper and Frieze [4] proved the existence of a constant h_0, such that if $h \geq h_0$, then graphs chosen uniformly at random from $L(n, K_h)$ and $L(n, K_{h,h})$ are aas Hamiltonian. Chebolu and Frieze [5] were able to expand this result to appropriately defined random lifts of complete directed graphs. The main result of this paper is as follows.

Theorem 1. *Let G be a graph with minimum degree at least five which contains at least two edge-disjoint Hamilton cycles whose union is not a bipartite graph. Then aas $\tilde{G} \in L_n(G)$ is Hamiltonian.*

This result, together with some of its straightforward generalizations, covers wide spectrum of graphs.

The structure of the paper is as follows. First we describe some general properties of random lifts and the idea behind the algorithm which finds the Hamilton cycle in $\tilde{G}$. Then we present the algorithm. In the next section we show that asymptotically almost surely it succeeds in finding a Hamilton cycle in $\tilde{G}$.

2. Preliminaries

We start with some general properties of random lifts that will be useful in proving the existence of Hamilton cycle in random lifts.

Lemma 2. *Let $h \geq 3$. Asymptotically almost surely a random n-lift of a cycle C_h on h vertices consists of a collection of at most $2 \log n$ disjoint cycles.*

Proof. If we remove one edge e from a cycle C_h, then we obtain a path. It is easy to see that the lift of the path P is a collection of n disjoint paths. Lifting the missing edge e is the same as matching at random the two sets of ends of those paths or connecting those ends according to some random permutation. The number of cycles created after joining those paths is then the same as the number of cycles in a random permutation on the set $[n] = \{1, 2, \ldots, n\}$. The precise distribution of the number of cycles in a random permutation is well known [7]. In particular aas the number of cycles in random permutation is smaller than $2 \log n$. $\square$

Our algorithm will be based on the path reversal technique of Pósa [13]. Let G be any connected graph and $P = v_0 v_1 \ldots v_m$ be a path in G. If $1 \leq i \leq m - 2$ and $\{v_m, v_i\}$ is an edge of G, then