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Finite 2-geodesic-transitive graphs of valency twice a prime

Wei Jin

School of Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, PR China
Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, PR China

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ABSTRACT

In this paper, we classify the family of connected 2-geodesic-transitive graphs of valency $2p$ where p is an odd prime.

1. Introduction

In this paper, graphs are finite, simple and undirected. In a non-complete graph Γ, a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$, and a 2-*geodesic* if in addition u, w are not adjacent. An *arc* is an ordered pair of adjacent vertices. The graph Γ is said to be 2-*arc-transitive* or 2-*geodesic-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on arcs, and also on 2-arcs or 2-geodesics, respectively. Clearly, every 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. The graph in Fig. 1 is the Kneser graph $KG_{6,2}$ which is 2-geodesic-transitive but not 2-arc-transitive with valency 6. Thus the family of non-complete 2-arc-transitive graphs is properly contained in the family of 2-geodesic-transitive graphs.

The first remarkable result about 2-arc-transitive graphs comes from Tutte [16,17], and this family of graphs has been studied extensively, see [10,12,13,15,18]. The local structure of the family of 2-geodesic-transitive graphs was determined in [3]. The papers [4,5] give classifications of all finite graphs which are 2-geodesic-transitive but not 2-arc-transitive, and which have valency 4 or prime valency, respectively. In this paper, we will give a classification of the family of 2-geodesic-transitive graphs of valency $2p$ where p is a prime.

For a vertex u of Γ, $\Gamma(u)$ denotes the set of vertices which are adjacent to u. The graph Γ is said to be *locally primitive* (*locally imprimitive*) if for every vertex u, the stabilizer A_u acts primitively

E-mail address: jinwei@jxufe.edu.cn.

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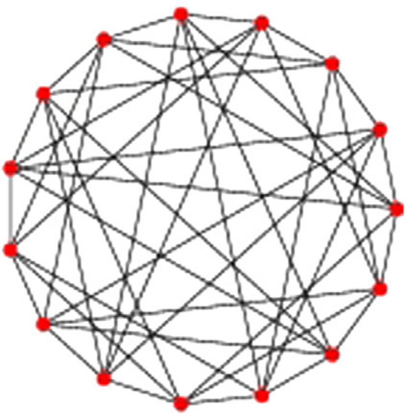

Fig. 1. Kneser graph $KG_{6,2}$.

(imprimitively) on $\Gamma(u)$ where $A := \mathrm{Aut}(\Gamma)$. A subgraph X of Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U. Let Λ be a finite graph. Then Γ is called *locally* Λ if for every $u \in V(\Gamma)$, $[\Gamma(u)] \cong \Lambda$. For two positive integers m, r, we denote by $\mathcal{F}(m, r)$ the family of all connected locally $m\mathrm{K}_r$ graphs. The connection between the family of graphs in $\mathcal{F}(m, r)$ and partial linear spaces was studied in [3]. The *line graph* $L(\Gamma)$ of Γ is the graph whose vertex set is the edge set of Γ, and any two distinct vertices are adjacent if and only if they have a common vertex in Γ.

Theorem 1.1. *Let Γ be a non-complete connected 2-geodesic-transitive graph of valency 2p where p is an odd prime. Let $A = \mathrm{Aut}(\Gamma)$ and $u \in V(\Gamma)$. Then one of the following holds.*

(1) *Γ is locally primitive of girth 3, and Γ is one of the following graphs: the halved 5-cube, the complement of the triangular graph $T(7)$, the Conway–Smith graph or the Hall graph.*

(2) *Γ is locally imprimitive of girth 3, and $\Gamma \in \{\mathrm{K}_{3[p]}, \mathrm{K}_{(p+1)[2]}\}$, or $\Gamma \in \mathcal{F}(p, 2)$, or one of the following is true.*

(2.1) *Γ is a line graph and $[\Gamma(u)] \cong \mathrm{K}_2 \square \mathrm{K}_p$.*

(2.2) *Γ is a line graph, A_u has two blocks of cardinality p in $\Gamma(u)$ but does not have blocks of cardinality 2, and the subgraph induced by a block is isomorphic to K_p.*

(2.3) *A_u has p blocks, $\Delta_i = \{v_i, v_i'\}$, $i = 1, \ldots, p$, in $\Gamma(u)$ but does not have blocks of cardinality p, $\Sigma := [\Gamma(u)]$ is connected and $[\Delta_i] \cong \mathrm{K}_2$. Either $[\Delta_i \cup \Delta_j] \cong C_4$ whenever $i \neq j$, $|\Sigma(v_i)| = p$ and $\Sigma(v_i) = \Sigma_2(v_i') \cup \{v_i'\}$; or $\Sigma \cong \widehat{\Sigma}[\mathrm{K}_2]$, where $\widehat{\Sigma}$ as in Definition 3.6, is a vertex-transitive graph of p vertices with valency $2(p-1)/3$ or $(p-1)/2$.*

(3) *Γ has girth at least 4 and is 2-arc-transitive.*

The graphs in Theorem 1.1 are defined in Sections 2 and 3.

Remark 1.2. (1) For any prime p, there exist graphs in the family $\mathcal{F}(p, 2)$. For instance, the Hamming graph $\mathrm{H}(p, 3)$ (with vertex set $\mathbb{Z}_3^p = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$, where $\mathbb{Z}_3 = \{0, 1, 2\}$ is the ring of integers modulo 3, and two vertices u, v are adjacent if and only if $u - v$ has exactly one non-zero entry) is in $\mathcal{F}(p, 2)$, and it is also 2-geodesic-transitive (see [6, Proposition 2.2]).

(2) The Kneser graph $KG_{6,2}$ belongs to the class $\mathcal{F}(3, 2)$.

(3) Suppose that Γ is a line graph in Theorem 1.1(2.2), A_u has two blocks Δ_1, Δ_2 of cardinality p in $\Gamma(u)$, and $[\Delta_1] \cong \mathrm{K}_p \cong [\Delta_2]$. Then $|\Gamma(u) \cap \Gamma(v) \cap \Delta_i| < p - 1$ where $v \in \Delta_j$ and $\{\Delta_i, \Delta_j\} = \{\Delta_1, \Delta_2\}$.

(4) Let $\Gamma = J(p+2, p)$ where p is an odd prime. Then $[\Gamma(u)] \cong \mathrm{K}_2 \square \mathrm{K}_p$, and by [6, Proposition 2.1], Γ is 2-geodesic-transitive, so Γ is in Theorem 1.1(2.1). (Let $\Omega = \{1, 2, \ldots, n\}$ where $n \geq 3$, and let $1 \leq k \leq [\frac{n}{2}]$ where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Then the *Johnson graph* $J(n, k)$ is the graph whose vertex set is the set of all k-subsets of Ω, and two k-subsets u and v are adjacent if and only if $|u \cap v| = k - 1$.)