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On judicious partitions of hypergraphs with edges of size at most 3

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ABSTRACT

Bollobás and Scott (2002) conjectured that a hypergraph with m_i edges of size i for $i = 1, \cdots, k$ has a bipartition in which each vertex class meets at least $m_1/2+3m_2/4+\cdots+(1-1/2^k)m_k+o(m)$ edges where $m = \sum_{i=1}^{k} m_i$. For the case $k = 2$, this conjecture has been proved by Ma et al. (2010). In this paper, we consider this conjecture for the case $k = 3$. In fact, we prove that a hypergraph with m_i edges of size i for $i = 1, 2, 3$ has a bipartition in which each vertex class meets at least $m_1/2 + 3m_2/4 + 23m_3/27 + o(m)$ edges where $m = m_1 + m_2 + m_3$.

1. Introduction

Let G be a graph or hypergraph, and let $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. We write $e(S) := |\{e \in E(G) : e \subseteq S\}|$, $e(S, T) := |\{e \in E(G) : e \cap S \neq \emptyset \text{ and } e \cap T \neq \emptyset\}|$, $d(S) := |\{e \in E(G) : e \cap S \neq \emptyset\}|$.

Many classical graph partitioning problems seek partitions of a graph that minimize or maximize a single quantity. For example, the Max-Cut Problem asks for a vertex bipartition V_1, V_2 of a graph G which maximizes $e(V_1, V_2)$.

Problems that ask for partitions of a graph to optimize several quantities simultaneously are called judicious partitioning problems. One such problem is the Bottleneck Bipartition Problem: given a graph G, find a bipartition V_1, V_2 of $V(G)$ which minimizes $\max\{e(V_1), e(V_2)\}$. This problem is NP-hard, see [10]. Note that in the Max-Cut Problem, we seek a partition in which $e(V_1, V_2)$ is large, or equivalently $e(V_1) + e(V_2)$ is small, but we do not care how edges are shared between V_1 and V_2. In the Bottleneck Bipartition Problem, we ask for a partition where $e(V_1)$ and $e(V_2)$ are small simultaneously.

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It is shown that the Max-Cut Problem and the Bottleneck Bipartition Problem are related [1]. For more results and other interesting partitioning problems we refer the reader to [5,9].

For hypergraphs, there are analogous judicious partitioning problems which seem to be much more difficult. One of such problems is the following conjecture proposed by Bollobás and Scott (see [3,5]).

Conjecture 1. *Let $r \geqslant 3$ and $k \geqslant 2$ be fixed integers. Then every r-uniform hypergraph G with m edges has a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that for $i = 1, \ldots, k$,*

$$e(V_i) \leqslant \frac{m}{k^r} + o(m).$$

This conjecture holds for $r = 3$ [3].

Theorem 2. *For $k \geqslant 2$, every 3-uniform hypergraph G has a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that for $i = 1, \ldots, k$,*

$$e(V_i) \leqslant \frac{m}{k^3} + o(m).$$

Particularly, every 3-uniform hypergraph G has a bipartition $V(G) = V_1 \cup V_2$ such that for $i = 1, 2$, $e(V_i) \leqslant m/8 + o(m)$.

In this paper we consider the judicious partitioning problems of hypergraphs with m_i edges of size i, $i = 1, \ldots, k$, which are called mixed partition problems [5].

Bollobás and Scott [4] proved the following result.

Theorem 3. *Let G be a hypergraph with m_i edges of size i, $i = 1, \ldots, k$. Then there is a partition of $V(G)$ into sets V_1, V_2 such that for $i = 1, 2$,*

$$d(V_i) \geqslant \frac{m_1 - 1}{3} + \frac{2m_2}{3} + \cdots + \frac{km_k}{k+1}.$$

Then they [4] suggested that the term $(m_1 - 1)/3$ might be replaced by $(m_1 - 1)/2$.

Conjecture 4. *For any hypergraph G with m_i edges of size i, $i = 1, \ldots, k$, there is a partition of $V(G)$ into sets V_1, V_2 such that for $i = 1, 2$,*

$$d(V_i) \geqslant \frac{m_1 - 1}{2} + \frac{2m_2}{3} + \cdots + \frac{km_k}{k+1}.$$

Haslegrave [6] proved that every hypergraph with m_i edges of size i, $i = 1, 2$ has a partition into sets V_1, V_2 such that for $i = 1, 2$, $d(V_i) \geqslant \frac{m_1-1}{2} + \frac{2m_2}{3}$.

Moreover, Bollobás and Scott [5] conjectured that asymptotically the bound might be improved. In fact, they proposed the following conjecture.

Conjecture 5. *Every hypergraph with m_i edges of size i for $i = 1, \ldots, k$ has a bipartition into V_1, V_2 such that for $i = 1, 2$,*

$$d(V_i) \geqslant \frac{m_1}{2} + \frac{3m_2}{4} + \frac{7m_3}{8} + \cdots + \left(1 - \frac{1}{2^k}\right) m_k + o(m)$$

where $m = m_1 + \cdots + m_k$.

Note that if we take a random partition V_1, V_2, then $\mathbb{E}(d(V_i)) = \frac{m_1}{2} + \frac{3m_2}{4} + \cdots + (1 - \frac{1}{2^k})m_k$. Note that for a bipartition, $d(V_i) = m - e(V_{3-i})$ for $i = 1, 2$. For the case $k = 2$, this conjecture has been proved by Ma, Yen and Yu [8]. In this paper, we consider this conjecture for the case $k = 3$. In fact, we prove the following result.