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Cubicity, degeneracy, and crossing number

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ABSTRACT

A k-box $B = (R_1, \ldots, R_k)$, where each R_i is a closed interval on the real line, is defined to be the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$. If each R_i is a unit-length interval, we call B a k-cube. The *boxicity* of a graph G, denoted as box(G), is the minimum integer k such that G is an intersection graph of k-boxes. Similarly, the *cubicity* of G, denoted as cub(G), is the minimum integer k such that G is an intersection graph of k-cubes.

It was shown in [L. Sunil Chandran, Mathew C. Francis, Naveen Sivadasan. Cubicity and bandwidth. Graphs and Combinatorics 29 (1) (2013) 45–69] that, for a graph G with n vertices and maximum degree Δ, $\mathrm{cub}(G) \leq \lceil 4(\Delta + 1) \log n \rceil$. In this paper we show the following:

- For a k-degenerate graph G, $\mathrm{cub}(G) \leq (k + 2)\lceil 2e \log n \rceil$. This bound is tight up to a constant factor.

Since k is at most Δ and can be much lower, this clearly is an asymptotically stronger result. Moreover, we have an efficient deterministic algorithm that runs in $O(n^2k)$ time to output an $O(k \log n)$-dimensional cube representation for G. The above result has the following interesting consequences:

- If the crossing number of a graph G is t, then box(G) is $O(t^{\frac{1}{4}} \lceil \log t \rceil^{\frac{3}{4}})$. This bound is tight up to a factor of $O((\log t)^{\frac{1}{4}})$. We also show that if G has n vertices, then cub(G) is $O(\log n + t^{1/4} \log t)$.
- Let dim($\mathcal{P}$) denote the *poset dimension* of a partially ordered set $(\mathcal{P}, \leq)$. We show that $\dim(\mathcal{P}) \leq 2(k + 2)\lceil 2e \log n \rceil$, where k denotes the degeneracy of the underlying comparability graph of $\mathcal{P}$.
- We show that the cubicity of almost all graphs in the $\mathcal{G}(n, m)$ model is $O(d_{av} \log n)$, where $d_{av} = \frac{2m}{n}$ denotes the average degree of the graph under consideration.

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1. Introduction

A graph G is an *intersection graph* of sets from a family of sets $\mathcal{F}$ if there exists a map $f : V(G) \to \mathcal{F}$ such that $(u, v) \in E(G) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. The representation of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. Probably the most well studied class of intersection graphs is the class of *interval graphs*. Interval graphs are the intersection graphs of closed intervals on the real line. The class of *indifference graphs* or *unit interval graphs* is a class of restricted forms of interval graphs that allow only intervals of unit length.

An interval on the real line can be generalized to a "k-box" in $\mathbb{R}^k$. A *k-box* $B = (R_1, \ldots, R_k)$, where each R_i is a closed interval on the real line, is defined to be the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$. If each R_i is a unit-length interval, we call B a *k-cube*. Thus, 1-boxes are just closed intervals on the real line whereas 2-boxes are axis-parallel rectangles in the plane. The parameter *boxicity* of a graph G, denoted as $\text{box}(G)$, is the minimum integer k such that G is an intersection graph of k-boxes. Similarly, the *cubicity* of G, denoted as $\text{cub}(G)$, is the minimum integer k such that G is an intersection graph of k-cubes. Thus, interval graphs are the graphs with boxicity equal to 1 and unit interval graphs are the graphs with cubicity equal to 1. A *k-box representation* or a *k-dimensional box representation* of a graph G is a mapping of the vertices of G to k-boxes such that two vertices in G are adjacent if and only if their corresponding k-boxes have a non-empty intersection. In a similar way, we define the *k-cube representation* (or *k-dimensional cube representation*) of a graph G. Since k-cubes (by definition) are also k-boxes, the boxicity of a graph is at most its cubicity.

The concepts of boxicity and cubicity were introduced by F.S. Roberts in 1969 [19]. Roberts showed that for any graph G on n vertices, $\text{box}(G) \leq \lfloor \frac{n}{2} \rfloor$ and $\text{cub}(G) \leq \lfloor \frac{2n}{3} \rfloor$. Both bounds are tight since $\text{box}(K_{2,2,\ldots,2}) = \lfloor \frac{n}{2} \rfloor$ and $\text{cub}(K_{3,3,\ldots,3}) = \lfloor \frac{2n}{3} \rfloor$ where $K_{2,2,\ldots,2}$ denotes the complete $n/2$-partite graph with two vertices in each part and $K_{3,3,\ldots,3}$ denotes the complete $n/3$-partite graph with three vertices in each part. It is easy to see that the boxicity of any graph is at least the boxicity of any induced subgraph of it.

Box representation of graphs finds application in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research (see [10]). Given a low dimensional box representation, some well known NP-hard problems become polynomial time solvable. For instance, the max-clique problem is polynomial time solvable for graphs with boxicity k because the number of maximal cliques in such graphs is only $O((2n)^k)$.

1.1. Previous results on boxicity and cubicity

It was shown by Cozzens [9] that computing the boxicity of a graph is **NP**-hard. Kratochvíl [14] showed that deciding whether the boxicity of a graph is at most 2 itself is **NP**-complete. It has been shown by Yannakakis [23] that deciding whether the cubicity of a given graph is at least 3 is **NP**-hard.

Researchers have tried to bound the boxicity and cubicity of graph classes with special structure. Scheinerman [20] showed that the boxicity of outerplanar graphs is at most 2. Thomassen [21] proved that the boxicity of planar graphs is bounded from above by 3. Upper bounds for the boxicity of many other graph classes such as chordal graphs, AT-free graphs, permutation graphs etc. were shown in [8] by relating the boxicity of a graph with its treewidth. The cube representations of special classes of graphs like hypercubes and complete multipartite graphs were investigated in [19,15,16].

Various other upper bounds on boxicity and cubicity in terms of graph parameters such as maximum degree, treewidth etc. can be seen in [4,5,3,12,8]. The ratio of cubicity to boxicity for any graph on n vertices was shown to be at most $\lceil \log_2 n \rceil$ in [6].

1.2. Equivalent definitions for boxicity and cubicity

Let G and $G_1, \ldots, G_b$ be graphs such that $V(G_i) = V(G)$ for $1 \leq i \leq b$. We say that $G = \bigcap_{i=1}^{b} G_i$ when $E(G) = \bigcap_{i=1}^{b} E(G_i)$. Below, we state two very useful lemmas due to Roberts [19].