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On symmetric quadrangulations and triangulations

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ABSTRACT

This article presents new enumerative results related to symmetric planar maps. In the first part a new way of enumerating rooted simple quadrangulations and rooted simple triangulations is presented, based on the description of two different quotient operations on symmetric simple quadrangulations and triangulations. In the second part, based on the results of Bouttier, Di Francesco and Guitter and on quotient and substitution operations, the series of three families of symmetric quadrangular and triangular dissections of polygons are computed, with control on the distance from the central vertex to the outer boundary.

0. Introduction

Enumeration of families of plane maps, that is, plane embeddings of graphs, has received a lot of attention since 1960s; several methods can be applied: the recursive method introduced by Tutte [15], the random matrix method introduced by Brézin et al. [4], and the bijective method introduced by Cori and Vauquelin [7] and Schaeffer [12]. In the first part of this note, we show another method for the enumeration of rooted simple quadrangulations and triangulations based on quotienting symmetric simple versions of them. Historically, the enumeration of symmetric maps of order k (*i.e.*, such that a rotation of order k fixes the map) was reduced to the enumeration of rooted maps via a quotient argument, a method used by Liskovets [10]. We proceed in the reverse way, namely we use two quotient operations on symmetric simple quadrangulations and triangulations to build in each case an algebraico-differential equation (Eqs. (3) and (8)) satisfied by the generating series of rooted corresponding simple maps, which can be explicitly solved to obtain the formulas for the number of rooted simple quadrangulations (due to Tutte [15] and bijectively proved by Schaeffer [12])

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and of rooted simple triangulations (due to Tutte [14] and bijectively proved by Poulalhon and Schaeffer [11]). One quotient operation is classical and is described in Section 1; the other quotient operation is new and, as described in Section 2, relies deeply on the existence and properties of α-orientations. The new equations for generating series of simple quadrangulations and triangulations are derived and solved in Section 3.

The results in the second part are expressions of the series of several families of symmetric quadrangular and triangular dissections with control on the distance from the central vertex to the outer boundary. We recall that symmetric dissections have been counted according to the number of inner faces by Brown [5,6] using the recursive method (Liskovet's quotient method [10] can also be applied, reducing the enumeration to rooted quadrangular dissections). Our approach, developed in Sections 4 and 5, relies on the quotient method and substitution operations combined with results by Bouttier et al. [1,3], which express the series of quadrangulations or triangulations with a marked vertex and marked edge at prescribed distance from each other. Our expressions illustrate again the property that the series expression of a "well behaved" map family $\mathcal{M}$ refined by a distance parameter d is typically expressed in terms of the dth power of an algebraic series of singularity type $z^{1/4}$ (implying that asymptotically the distance parameter d on a random map of size n in $\mathcal{M}$ converges in the scale $n^{1/4}$ as a random variable).

1. Plane maps, symmetry and classical quotient

1.1. Triangulations, quadrangulations and dissections

A *plane map* is a connected graph embedded in the plane up to continuous deformation; the unique unbounded face of a plane map is called the *outer face*, and the other ones are called *inner faces*. Vertices and edges are also called outer if they belong to the outer face and inner otherwise. A map is said to be *rooted* if an edge of the outer face is marked and oriented so as to have the outer face on its left. This edge is the *root edge*, and its origin is the *root vertex*. A map is *pointed* if one of its *inner* vertices is marked. For any map M, we denote by $\mathcal{V}(M)$, $\mathcal{F}(M)$, $\mathcal{E}(M)$ its sets of vertices, faces and edges, and by $v(M)$, $f(M)$, $e(M)$ their cardinalities.

Triangulations and quadrangulations are respectively maps with all faces of degree 3 or 4, and to avoid the degenerated cases, the outer face is required to be a simple cycle. For $k \geqslant 1$ and $d \geqslant 3$, a *d-angular dissection of a k-gon* or *d-angular k-dissection* is a map whose outer face contour is a simple cycle of length k, with all inner faces of one and the same degree d. A dissection is said to be *triangular* if d equals 3 and *quadrangular* if d equals 4. Observe that quadrangular k-dissections can only exist for even k.

A map is said to be *simple* if it has no multiple edges; a d-angular k-dissection is called *irreducible* if the interior of every cycle of length at most d is a face.

1.2. Symmetric maps and classical quotient

For $k \geqslant 2$, a dissection D is said to be *k-symmetric* if its plane embedding (conveniently deformed) is invariant by a $2\pi/k$-rotation centered at a vertex — called the *center* of D.

As observed by Liskovets [10], any two semi-infinite straight lines starting from the center and forming an angle of $2\pi/k$ delimit a sector of D. When keeping only this sector and pasting these two lines together, we obtain a plane map, called the *k-quotient map* of D. In other words, the $2\pi/k$-rotation defines equivalence relations on the sets $\mathcal{V}(D)$ and $\mathcal{E}(D)$, and the quotient map of D is the map in which equivalent vertices and equivalent edges are identified. Fig. 1 shows the example of two symmetric dissections of a hexagon and their quotients. Denote by $o(D)$ the degree of the outer face of a dissection. The following lemma is straightforward.

Lemma 1.1. *For $k \geqslant 2$, let D be a k-symmetric dissection, and E its k-quotient; we have*

$$v(D) - 1 = k\,(v(E) - 1), \qquad e(D) = k\,e(E),$$
$$f(D) - 1 = k\,(f(E) - 1), \qquad o(D) = k\,o(E).$$