European Journal of Combinatorics 35 (2014) 67–74

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Infinite reduction of divisors on metric graphs

Spencer Backman

School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332-0160, USA

ARTICLE INFO

Article history:
Available online 31 July 2013

ABSTRACT

We demonstrate that the greedy algorithm for reduction of divisors on metric graphs need not terminate by modeling the Euclidean algorithm in this context. We observe that any infinite reduction has a well defined limit, allowing us to treat the greedy reduction algorithm as a transfinite algorithm and to analyze its running time via ordinal numbers. Matching lower and upper bounds on worst case running time of $O(\omega^n)$ are provided.

1. Introduction

Chip-firing on graphs has been studied in various contexts for over 20 years. The theory has found new applications in the recent work of Baker and Norine [5], who showed that by studying chip-firing, one may extend the work of Bacher, de la Harpe, and Nagnibeda [2] on the theory of linear equivalence of divisors on graphs. In particular, they were able to demonstrate the existence of a Riemann–Roch theorem for graphs analogous to the classical statement for curves. Gathmann and Kerber [8], and independently Mikhalkin and Zharkov [13], proved a Riemann–Roch theorem for tropical curves. The approach of Gathmann and Kerber was to establish the tropical Riemann–Roch theorem as a limit of Baker and Norine's result for graphs under subdivision of edges. Hladky, Kral, and Norine [10] then showed that this theorem may be proven in an elementary way by studying the combinatorics of chip-firing on abstract tropical curves, i.e., metric graphs. Several papers have pursued this approach further along with other consequences for the theory of linear equivalence of divisors on tropical curves [1,7, 9,12].

The central combinatorial objects in this study, for both graphs and tropical curves, are the so-called q-reduced divisors (known elsewhere in the literature as superstable configurations or G-parking functions). A q-reduced divisor is a special representative from the class of divisors linearly equivalent to a given divisor. There is an algorithmic method for obtaining the unique q-reduced divisor consisting of two parts. In this paper, we investigate the second, more subtle part of this process known

E-mail addresses: spencerbackman@gmail.com, sbackman@math.gatech.edu.

0195-6698/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ejc.2013.06.024

as reduction. We offer a new short proof of Luo's result that Dhar's reduction algorithm terminates after a finite number of iterations. We then investigate the greedy reduction algorithm, which in the graphical case is known to succeed. We show that the Euclidean algorithm may be modeled by the greedy reduction of divisors on metric graphs. By evaluating this algorithm on two incommensurable numbers, we obtain a run of the greedy reduction algorithm which does not terminate.

After observing that any infinite reduction has a well-defined limit, we analyze the running time of the greedy algorithm via ordinal numbers. We demonstrate matching upper and lower bounds on worst case running time of $O(\omega^n)$. The lower bound is obtained by gluing n copies of the Euclidean algorithm example together and ordering the firings lexicographically. The upper bound of $\omega^{\deg(D)}$ is provided by an inductive argument.

2. Metric chip-firing and reduced divisors

A metric graph Γ is a metric space which can be obtained from an edge weighted graph G by viewing each edge with weight $w_{i,j}$ as being isometric to an interval of length $w_{i,j}$. Each point interior to an edge has a neighborhood homeomorphic to an open interval and each vertex has a small neighborhood homeomorphic to a star. The *degree* of a point $p \in \Gamma$ is the number of tangent directions at p. A vertex is called a *combinatorial vertex* if it has degree other than 2.

This paper concerns certain combinatorial aspects of chip-firing on metric graphs, so we will take a rather concrete working definition of chip-firing. For the sake of completeness, we begin with a slightly more abstract definition. Fix a metric graph Γ and a parameterization of the edges of Γ. Let f be a piecewise affine function with integer slopes on Γ. We define the Laplacian operator Q applied to f at a point to be the sum of the slopes of the function as we approach p along each of the tangent directions at p. We note that $Q(f)(p) = 0$ if f is differentiable at p. We define a *divisor* D on Γ to be a formal sum of points from Γ with integer coefficients, all but a finite number of which are zero. We say that D has $D(p)$ *chips* at p. Given some divisor D on Γ, we define the chip-firing operation f applied to D to be $D - Q(f)$. We say that two divisors are linearly equivalent if they differ by some chip-firing move. A divisor E is said to be *effective* if it has a nonnegative number of chips at each point.

We now give the definition of chip-firing on metric graphs which will be used for the remainder of the paper. Let X and Y be two disjoint open connected subsets of Γ such that the $\Gamma \setminus (X \cup Y) = Z$ is isometric to a disjoint collection of closed intervals of length ϵ. Note that the set Z defines a minimal cut in Γ. Now, we define the divisor $Q(f)$ as the divisor which is negative one at the end points of these intervals on the boundary of X and positive one at the endpoints on the boundary of Y. One may intuitively understand this divisor as pushing a chip along each edge in this cut a fixed distance ϵ. We take this to be the basic type of chip-firing move and call ϵ the *length* of the firing. Note that the chip-firing divisor is of the form $Q(f)$ where f is the piecewise affine function with integer slopes which is 0 on X, ϵ on Y, and has slope 1 on the each open interval in Z. We write $\epsilon(f)$ for the length of the firing f. As is noted in [3], any piecewise affine function with integer slopes can be expressed as a finite sum of the functions just described, so we will not sacrifice any generality by restricting our definition of chip-firing to be basic chip-firing moves.

A *q-reduced divisor* is a divisor which is nonnegative at each point other than $q \in \Gamma$, such that any firing $Q(f)$ which pushes chips toward q causes some point to go into debt. It is proven in [10] that given any divisor D on a metric graph Γ, there exists a unique q-reduced divisor ν which can be reached from D by a sequence of chip-firing moves. Moreover, there exists an effective divisor E equivalent to D if and only if ν is effective. An algorithmic way of obtaining such a divisor was described by Luo [12]. His method is to first bring every point other than q out of debt by some sequence of chip-firing moves. Once we have obtained such a configuration, we may perform firings which push chips back toward q without causing any vertex to go into debt. We call this second part of the process *reduction*. Luo's method for reducing a divisor is to use a generalization of Dhar's burning algorithm originally investigated in the study of the sandpile model.

Dhar's burning algorithm may be described in the following informal way: Let D be a divisor which is nonnegative at every point of Γ other than q. Place $D(p)$ firefighters at each point p other that q. Light a fire at q and let the fire spread through Γ along the edges. Every time the fire reaches a firefighter, it stops. If the fire approaches a point from more directions than there are firefighters present, these