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# The maximum sum and the maximum product of sizes of cross-intersecting families

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ABSTRACT

We say that a set $A$ $t$-*intersects* a set $B$ if $A$ and $B$ have at least $t$ common elements. A family $\mathcal{A}$ of sets is said to be $t$-*intersecting* if each set in $\mathcal{A}$ $t$-intersects all the other sets in $\mathcal{A}$. Families $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ are said to be *cross-$t$-intersecting* if for any $i$ and $j$ in $\{1, 2, \dots, k\}$ with $i \neq j$, every set in $\mathcal{A}_i$ $t$-intersects every set in $\mathcal{A}_j$. We prove that for any finite family $\mathcal{F}$ that has at least one set of size at least $t$, there exists an integer $\kappa \leq |\mathcal{F}|$ such that for any $k \geq \kappa$, both the sum and the product of sizes of $k$ cross-$t$-intersecting subfamilies $\mathcal{A}_1, \dots, \mathcal{A}_k$ (not necessarily distinct or non-empty) of $\mathcal{F}$ are maxima if $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{L}$ for some largest $t$-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$. We then study the smallest possible value of $\kappa$ and investigate the case $k < \kappa$; this includes a cross-intersection result for straight lines that demonstrates that it is possible to have $\mathcal{F}$ and $\kappa$ such that for any $k < \kappa$, the configuration $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{L}$ is neither optimal for the sum nor optimal for the product. We also outline solutions for various important families $\mathcal{F}$, and we provide solutions for the case when $\mathcal{F}$ is a power set.


## 1. Introduction

Unless otherwise stated, throughout this paper we shall use small letters such as $x$ to denote positive integers or elements of a set, capital letters such as $X$ to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose elements are sets themselves). Unless specified, sets and families are taken to be finite and may be the *empty set* $\emptyset$. An $r$-*set* is a set of size $r$, that is, a set having exactly $r$ elements. For any integer $n \geq 1$, $[n]$ denotes the set $\{1, \dots, n\}$ of the first $n$ positive integers.

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Given an integer $t \geq 1$, we say that a set $A$ *$t$-intersects* a set $B$ if $A$ and $B$ have at least $t$ common elements. A family $\mathcal{A}$ is said to be *$t$-intersecting* if each set in $\mathcal{A}$ $t$-intersects all the other sets in $\mathcal{A}$ (i.e. $|A \cap B| \geq t$ for any $A, B \in \mathcal{A}$ with $A \neq B$). A 1-intersecting family is also simply called an *intersecting family*. Families $\mathcal{A}_1, \dots, \mathcal{A}_k$ are said to be *cross-$t$-intersecting* if for any $i$ and $j$ in $[k]$ with $i \neq j$, every set in $\mathcal{A}_i$ $t$-intersects every set in $\mathcal{A}_j$ (i.e. $|A \cap B| \geq t$ for any $A \in \mathcal{A}_i$ and any $B \in \mathcal{A}_j$). Cross-1-intersecting families are also simply called *cross-intersecting families*.

Let $\binom{[n]}{r}$ denote the family of all subsets of $[n]$ of size $r$. The classical Erdős–Ko–Rado (EKR) Theorem [17] says that if $n$ is sufficiently larger than $r$, then the size of any $t$-intersecting subfamily of $\binom{[n]}{r}$ is at most $\binom{n-t}{r-t}$, which is the number of sets in the $t$-intersecting subfamily of $\binom{[n]}{r}$ consisting of those sets having $[t]$ as a subset. The EKR Theorem inspired a wealth of results of this kind, that is, results that establish how large a system of sets can be under certain intersection conditions; see [10,14,18].

For $t$-intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. For cross-$t$-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-$t$-intersecting families (note that the product of sizes of $k$ families $\mathcal{A}_1, \dots, \mathcal{A}_k$ is the number of $k$-tuples $(A_1, \dots, A_k)$ such that $A_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximising the sum or the product of sizes of $k$ cross-$t$-intersecting subfamilies (not necessarily distinct or non-empty) of a given family $\mathcal{F}$.

The main result in this paper (Theorem 1.1 below) relates both the maximum sum and the maximum product of sizes of $k$ cross-$t$-intersecting subfamilies of any family $\mathcal{F}$ to the maximum size of a $t$-intersecting subfamily of $\mathcal{F}$ when $k$ is not smaller than a certain value depending on $\mathcal{F}$ and $t$. It gives the maximum sum and the maximum product in terms of the size of a largest $t$-intersecting subfamily.

For any non-empty family $\mathcal{F}$, let $\alpha(\mathcal{F})$ denote the size of a largest set in $\mathcal{F}$ (i.e. $\alpha(\mathcal{F}) = \max\{|F|: F \in \mathcal{F}\}$). Suppose $\alpha(\mathcal{F}) < t$, and let $\mathcal{A}_1, \dots, \mathcal{A}_k$ ($k \geq 2$) be subfamilies of $\mathcal{F}$. Then $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-$t$-intersecting if and only if at most one of them is non-empty (since no set in $\mathcal{F}$ $t$-intersects itself or another set in $\mathcal{F}$). Thus, if $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-$t$-intersecting, then the product of their sizes is 0 and the sum of their sizes is at most the size $|\mathcal{F}|$ of $\mathcal{F}$ (which is attained if and only if one of them is $\mathcal{F}$ and the others are all empty). This completely solves our problem for the case $\alpha(\mathcal{F}) < t$.

We now address the case $\alpha(\mathcal{F}) \geq t$. Before stating our main result, we need to introduce some definitions and parameters.

For any family $\mathcal{A}$, let $\mathcal{A}^{t,+}$ be the ($t$-intersecting) subfamily of $\mathcal{A}$ given by

$$\mathcal{A}^{t,+} = \{A \in \mathcal{A}: |A \cap B| \geq t \text{ for any } B \in \mathcal{A} \text{ such that } A \neq B\},$$

and let

$$\mathcal{A}^{t,-} = \mathcal{A} \setminus \mathcal{A}^{t,+}.$$

In simple terms, a set $A$ in $\mathcal{A}$ is in $\mathcal{A}^{t,-}$ if there exists a set $B$ in $\mathcal{A}$ such that $A \neq B$ and $A$ does not $t$-intersect $B$, otherwise $A$ is in $\mathcal{A}^{t,+}$. The definitions of $\mathcal{A}^{t,+}$ and $\mathcal{A}^{t,-}$ are generalisations of the definitions of $\mathcal{A}^*$ and $\mathcal{A}'$ in [5,8,9,11,12], respectively; $\mathcal{A}^* = \mathcal{A}^{1,+}$ and $\mathcal{A}' = \mathcal{A}^{1,-}$.

Let $l(\mathcal{F}, t)$ denote the size of a largest $t$-intersecting subfamily of a non-empty family $\mathcal{F}$. For any subfamily $\mathcal{A}$ of $\mathcal{F}$, we define

$$\beta(\mathcal{F}, t, \mathcal{A}) = \begin{cases} \dfrac{l(\mathcal{F}, t) - |\mathcal{A}^{t,+}|}{|\mathcal{A}^{t,-}|} & \text{if } \mathcal{A}^{t,-} \neq \emptyset; \\[2ex] \dfrac{l(\mathcal{F}, t)}{|\mathcal{F}|} & \text{if } \mathcal{A}^{t,-} = \emptyset; \end{cases}$$

so $|\mathcal{A}^{t,+}| + \beta(\mathcal{F}, t, \mathcal{A})|\mathcal{A}^{t,-}| \leq l(\mathcal{F}, t)$ (even if $\mathcal{A}^{t,-} = \emptyset$, because $|\mathcal{A}^{t,+}| \leq l(\mathcal{F}, t)$ since $\mathcal{A}^{t,+}$ is $t$-intersecting). We now define

$$\beta(\mathcal{F}, t) = \min\{\beta(\mathcal{F}, t, \mathcal{A}): \mathcal{A} \subseteq \mathcal{F}\}.$$