European Journal of Combinatorics 35 (2014) 141–154

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The range of thresholds for diameter 2 in random Cayley graphs

Demetres Christofides [a], Klas Markström [b]

[a] School of Sciences, UCLan Cyprus, 7080 Pyla, Cyprus

[b] Department of Mathematics and Mathematical Statistics, Umeå University, 90187 Umeå, Sweden

ARTICLE INFO

Article history:
Available online 4 July 2013

ABSTRACT

Given a group G, the model $\mathcal{G}(G,p)$ denotes the probability space of all Cayley graphs of G where each element of the generating set is chosen independently at random with probability p.

Given a family of groups (G_k) and a $c \in \mathbb{R}_+$ we say that c is the threshold for diameter 2 for (G_k) if for any $\varepsilon > 0$ with high probability $\Gamma \in \mathcal{G}(G_k,p)$ has diameter greater than 2 if $p \leqslant \sqrt{(c-\varepsilon)\frac{\log n}{n}}$ and diameter at most 2 if $p \geqslant \sqrt{(c+\varepsilon)\frac{\log n}{n}}$. In Christofides and Markström (in press) [5] we proved that if c is a threshold for diameter 2 for a family of groups (G_k) then $c \in [1/4, 2]$ and provided two families of groups with thresholds 1/4 and 2 respectively.

In this paper we study the question of whether every $c \in [1/4, 2]$ is the threshold for diameter 2 for some family of groups. Rather surprisingly it turns out that the answer to this question is negative. We show that every $c \in [1/4, 4/3]$ is a threshold but a $c \in (4/3, 2]$ is a threshold if and only if it is of the form $4n/(3n-1)$ for some positive integer n.

1. Introduction

Let us begin by recalling that given a group G and a subset S of G, the Cayley graph $\Gamma = \Gamma(G;S)$ of G with respect to S has the elements of G as its vertex set and g and h are neighbours if and only if $hg^{-1} \in S$ or $gh^{-1} \in S$. We ignore any loops, so in particular, whether $1 \in S$ or not is immaterial. Observe for example that Γ is connected if and only if the set S generates the group G. Throughout the

E-mail addresses: christofidesdemetres@gmail.com (D. Christofides), klas.markstrom@math.umu.se (K. Markström).

0195-6698/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ejc.2013.06.030

paper, we will often refer to the set S as the generating set of the graph Γ irrespectively of whether it is a generating set for the group G or not.

The model $\mathcal{G}(G, p)$ is the probability space of all graphs $\Gamma(G; S)$ in which every element of G is assigned to the set S independently at random with probability p. We refer the reader to [4,5] for a discussion of some similarities and differences between this model and various other models of random graphs.

In this paper we continue the study of the diameter of random Cayley graphs that was initiated in [5]. Given a family of groups (G_k) of orders n_k with $n_k \to \infty$ as $k \to \infty$ and a $c \in \mathbb{R}_+$ we say that c is the threshold for diameter 2 for (G_k) if for any $\varepsilon > 0$ with high probability $\Gamma \in \mathcal{G}(G_k, p)$ has diameter greater than 2 if $p \leqslant \sqrt{(c-\varepsilon)\frac{\log n}{n}}$ and diameter at most 2 if $p \geqslant \sqrt{(c+\varepsilon)\frac{\log n}{n}}$. In [5] we proved that if c is a threshold for diameter 2 for a family of groups (G_k) then $c \in [1/4, 2]$. Moreover this is best possible as $1/4$ is the threshold for the family of symmetric groups S_n while 2 is the threshold for the family of products of two-cycles $(C_2)^n$. Here we can also mention that for a model for random Cayley digraphs the probability of having diameter 2 was studied in [7].

Given the above result it is natural to ask whether for every $c \in [1/4, 2]$ there is a family of groups for which it is a threshold for diameter 2. Rather surprisingly, it turns out that the answer is negative. The aim of this paper is to give a characterisation of the values of c that can appear as thresholds.

Theorem 1.1. *Let $c \in [1/4, 2]$. Then c is the threshold for diameter 2 for a family of groups (G_k) if and only if $c \in [1/4, 4/3] \cup \left\{\frac{4n}{3n-1} : n \in \mathbb{N}\right\}$.*

We break the proof of Theorem 1.1 into the following results.

Theorem 1.2. *Let $c \in [1/4, 1/2)$. Then c is the threshold for diameter 2 for the family $S_n \times C_m$, where $m = \left\lfloor n!^{\frac{2c}{1-2c}} \right\rfloor$.*

Theorem 1.3. *Let $c \in [1/2, 1)$. Then c is the threshold for diameter 2 for the family $G = (C_2)^n \times (C_m)$ where $m = \left\lfloor 2^{\frac{(1-c)n}{c}} \right\rfloor$.*

Theorem 1.4. *Let $c \in [1, 4/3)$. Then c is the threshold for diameter 2 for the family $G = (C_2)^n \times (D_{2m})$ where $m = \left\lfloor 2^{\frac{(4-3c)n}{3c}} \right\rfloor$ and D_{2m} is the dihedral group of order $2m$.*

Theorem 1.5. *Suppose $c > 4/3$ is the threshold for diameter 2 for some family of groups. Then $c \in \left\{\frac{4n}{3n-1} : n \in \mathbb{N}\right\}$. Moreover, given a positive integer n, $\frac{4n}{3n-1}$ is the threshold for diameter 2 for the family $((C_2)^k \times D_{4n})_k$.*

It is evident that Theorem 1.1 follows from Theorems 1.2–1.5. These results will be consequences of the more general Theorem 2.1 which enables us to determine the threshold for diameter 2 for a family of groups provided we have enough information about some specific structure of the groups of this family.

As we mentioned this looks like a surprising result as one would expect that by taking 'suitable combinations' of groups with different thresholds, one should be able to interpolate to get all possible values between the two thresholds. This is exactly what happens in Theorem 1.2. Symmetric groups have threshold $1/4$, cyclic groups have threshold $1/2$, and by taking a suitable direct product we can get any threshold in the interval $[1/4, 1/2]$. However there is a natural reason why this result is not so surprising after all. As one would expect the result depends on various group properties and there are many such properties for which you cannot interpolate. We are thinking here of properties which say that when the group is 'too close to being abelian' then it actually is abelian. For example it is a trivial observation that if more than half of the elements of a group are central (i.e. they commute with every other element) then the group is abelian and so all elements are central. (In fact the same holds even when more than a quarter of the elements are central.) In a similar manner, if more than $5/8$ of pairs of