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A connected subgraph maintaining high connectivity

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ABSTRACT

It was proved by Mader that, for every integer l, every k-connected graph of sufficiently large order contains a vertex set X of order precisely l such that $G - X$ is $(k-2)$-connected. This is no longer true if we require X to be connected, even for $l = 3$.

Motivated by this fact, we are trying to find an "obstruction" for k-connected graphs without such a connected subgraph. It turns out that the obstruction is an essentially 3-connected subgraph W such that $G - W$ is still highly connected. More precisely, our main result says the following.

For $k \geq 7$ and every k-connected graph G, either there exists a connected subgraph W of order 4 in G such that $G - W$ is $(k-2)$-connected, or else G contains an "essentially" 3-connected subgraph W, i.e., a subdivision of a 3-connected graph, such that $G - W$ is still highly connected—actually, $(k-6)$-connected.

This result can be compared to Mader's result (Mader, 2002) [5] which says that every k-connected graph G of sufficiently large order ($k \geq 4$) has a connected subgraph H of order exactly 4 such that $G - H$ is $(k-3)$-connected.

1. Introduction

Problems of the following type have been well-studied by many researchers; see [3,4,7].

Problem 1. Given positive integers k, l, d with $k \geq d$, is it true that every k-connected graph of sufficiently large order has a vertex set X of order exactly l such that $G - X$ is $(k-d)$-connected?

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In connection with the above question, (k, l)-critical graphs, which we will define later, are studied by many researchers. A k-connected graph G is said to be (k, l)-*critical* if for any vertex set W of order l with $k \geq l$, $G - W$ is $(k - l)$-connected. The main question concerning (k, l)-critical graphs is that of whether or not there are only finitely many (k, l)-critical graphs; see the survey by Mader [7].

A best possible result for the above problem would be that we can find a vertex set W of prescribed order such that $G - W$ is $(k - 1)$-connected, but this is not true as Mader has pointed out in [6]. On the other hand, Mader has proved (in the same paper) that every k-connected graph G of sufficiently large order contains a vertex set S of prescribed order such that $G - S$ is $(k - 2)$-connected. A natural question which arose from the above result is: "Could it be true if we require S to be connected?" Unfortunately, this is no longer true. In [6], Mader has pointed out that for every $k \geq 18$, there are infinitely many k-connected graphs such that for any connected subgraph W of order exactly 3, $G - W$ is not $(k - 2)$-connected. So we cannot even hope for the case $k = 3$. Let us observe that Mader [2] has proved that there are only finitely many $(k, 3)$-critical graphs.

On the other hand, Mader [5] has proved that every k-connected graph G of sufficiently large order has a connected subgraph H of order exactly 4 such that $G - H$ is $(k - 3)$-connected. Again the connectivity is best possible.

Our motivation comes from the above result. More precisely, we are led to the following question: When can a k-connected graph G have a connected subgraph W of order exactly 4 such that $G - W$ is $(k - 2)$-connected? Which graphs would be obstructions? It turns out that the obstructions are nearly 3-connected, i.e., a subdivision of a 3-connected graph. In addition, these obstructions are, in a sense, non-separating subgraphs. More precisely, for any obstruction W, $G - W$ is $(k - 6)$-connected. (Note that by subgraph, we do not mean an induced subgraph but we allow deleting edges as well.) To state our main result, we need some definitions and notation, but before that, let us give further motivation.

Thomassen [9] conjectured that every $(a + b + 1)$-connected graph can be decomposed into two parts A and B in such a way that A is a-connected and B is b-connected. It was shown by Thomassen himself [9] that if $b \leq 2$, then the conjecture is true. Even the case $b = 3$ is not settled yet, and the conjecture is wide open for $a, b \geq 3$. We would like to prove this conjecture for when $b = 3$, but we have failed. However since our obstructions are essentially 3-connected, and otherwise, we could delete a connected subgraph of order 4 in such a way that the connectivity of the resulting graph does not decrease by more than 2, our result may be the first step towards proving the conjecture for when $b = 3$.

At this point, we should mention the 3-connected case for related problems. It was conjectured in [8] that for every positive integer l, every 3-connected graph G of sufficiently large order has a connected subgraph W of order precisely l such that $G - W$ is 2-connected. This conjecture is verified for $l = 2$ in [10], for $l = 3$ in [8] and for $l = 4$ in [1]. The cases $l \geq 5$ are open. So as we see here, when a given graph G is 3-connected, a stronger conclusion may be true.

We need some definitions and notation to state our main result.

Let C_4 be a quadrilateral, i.e., a cycle of length 4. Let C_5 be a pentagon, i.e., a cycle of length 5. Also let W_4 be a graph which is isomorphic to $K_1 + C_4$ and let W_5 be a graph which is isomorphic to $K_1 + C_5$, where $A + B$ means the graph obtained from the disjoint union of the graphs A and B by adding all the possible edges between them. A subdivision of K_4 of order 5 can be regarded as a graph which can be obtained from W_4 by the deletion of one edge, which is adjacent to the central vertex of W_4. So, in the following argument, we call this graph "W_4^-". Similarly, we define "W_5^-", which is a graph obtained from W_5 by the deletion of one edge, which is adjacent to the central vertex of W_5 (W_5^- can be also regarded as a subdivision of W_4). Let K_4^- be the graph obtained from K_4 by deleting one edge. Also let K_4^* be the graph obtained from K_4 by subdividing two independent edges with one vertex (thus K_4^* consists of six vertices).

Our main theorem is the following.

Theorem 1. *Let k be an integer with $k \geq 7$. Suppose that G is k-connected. Then G contains a subgraph M which satisfies one of the following:*

(i) *M is a 3-connected graph of order at most 6.*
(ii) $M \cong W_4^-$.