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## Lift-contractions

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## ABSTRACT

We introduce and study a partial order on graphs—lift-contractions. A graph  $H$  is a lift-contraction of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of edge lifts and edge contractions. We give sufficient conditions for a connected graph to contain every  $n$ -vertex graph as a lift-contraction and describe the structure of graphs with an excluded lift-contraction.

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## 1. Introduction

All graphs in this paper are undirected, loopless, and without multiple edges (unless mentioned otherwise).  $V(G)$  and  $E(G)$  denote the vertex and edge set of a graph  $G$ , respectively. The *degree* of a vertex  $v \in V(G)$  is the number of edges incident with it.  $K_n$  is the complete graph on  $n$  vertices. Given an edge  $e$  of a graph  $G$ , the result of the *contraction* of  $e$  in  $G$  is the graph obtained by removing  $e$  from  $G$  and then identifying its endpoints to a single vertex  $v_e$ . For notions and notations not defined here, we refer the reader to the monograph [5].

Given two edges  $e_1 = \{x, x_1\}$  and  $e_2 = \{x, x_2\}$  of  $G$ , incident with the same vertex  $x$ , and such that  $x_1 \neq x_2$ , we define the *lift* of  $e_1$  and  $e_2$  in  $G$  as the graph obtained by removing  $e_1$  and  $e_2$  from  $G$  and then adding the edge  $\{x_1, x_2\}$ . If a contraction or edge lift creates multiple edges, we reduce their multiplicity to one and keep the graph simple.

*Partial orders.* The study of partial orders on graphs is one of the basic research avenues in graph theory. One of the most comprehensive studies of partial orders is the theory of Graph Minors by Robertson and Seymour [11] (see also the last chapter of [5]). A graph  $H$  is a *minor* of another graph  $G$  ( $H \leq_m G$ )

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if  $H$  can be obtained from  $G$  by a sequence of vertex deletions, edge removals, and edge contractions. Some more restricted graph containment relations than graph minors, like contractions [3] or induced minors [9] have also been studied.

Graph immersions form another partial order that has been considered in the literature [4]. A graph  $H$  is an *immersion* of  $G$  if  $H$  can be obtained from  $G$  by a sequence of vertex deletions, edge removals, and edge lifts. The last operation was introduced by Lovász under the name of *splitting* as a reduction method to maintain edge connectivity [8].

In this paper, we introduce and study lift-contractions. We say that a graph  $H$  is a *lift-contraction* of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of edge lifts and edge contractions. We also define lift-minors. We say that a graph  $H$  is a *lift-minor* of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of vertex and edge deletions, edge lifts and contractions.

Being a lift-contraction (lift-minor) is a partial relation between graphs and we denote it by  $H \leq_{lc} G$  ( $H \leq_{lm} G$ ). If a graph  $H$  can be obtained from  $G$  by a sequence of contractions, we say that  $H$  is a contraction of  $G$  and we denote this by  $H \leq_c G$ . Clearly,  $H \leq_c G \Rightarrow H \leq_{lc} G \Rightarrow H \leq_{lm} G$  and  $H \leq_m G \Rightarrow H \leq_{lm} G$ .

**Forcing complete graphs.** When studying a partial order  $\leq$  on graphs, it is interesting to know under what conditions on  $G$ , for a fixed graph  $H$ ,  $H \leq G$ . Kostochka [7] and Thomason [13] independently proved that if the average degree of  $G$  is at least  $cn\sqrt{\log n}$ , then  $G$  contains  $K_n$  as a minor (for some constant  $c > 0$ ). Bollobás [2] showed that if the average degree of  $G$  is at least  $cn^2$ , then  $G$  contains  $K_n$  as a topological minor<sup>1</sup> (for some constant  $c > 0$ ). Recently, DeVos, Dvořák, Fox, McDonald, Mohar and Scheide [4] proved that if the minimum degree of  $G$  is at least  $200n$ , then  $G$  contains  $K_n$  as an immersion. For all these three partial orders, containing  $K_n$  implies containing any  $n$ -vertex graph.

In this paper, we identify three conditions on a connected graph  $G$  that force any  $n$ -vertex graph as a lift-contraction of  $G$ .

**Theorem 1.1.** *There exists a constant  $c$  such that every connected graph  $G$  of treewidth at least  $c \cdot n^4$  contains every  $n$ -vertex graph as a lift-contraction.*

**Theorem 1.2.** *There exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every 2-connected graph of pathwidth at least  $f(n)$  contains every  $n$ -vertex graph as a lift-contraction.*

**Theorem 1.3.** *There exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every connected graph with at least  $f(n)$  vertices and minimum degree at least 3 contains every  $n$ -vertex graph as a lift-contraction.*

We note that none of the three conditions above is alone enough to force all  $n$ -vertex graphs as a lift or as a contraction. In order to see this, consider a complete graph  $K$  with an arbitrarily large number of vertices. Because an edge lift does not change the number of vertices, we cannot obtain a graph with fewer vertices than  $K$  by taking edge lifts only. Because contracting an edge in  $K$  yields a new complete graph, we cannot obtain any non-complete graph by performing edge contractions only.

**Structural theorem.** Another point of focus, when studying partial orders on graphs, is to understand the structure of nontrivial ideals in this order. The best known example is the structural theorem on graphs with an excluded minor by Robertson and Seymour [11]. Recently, a structural description of graphs with an excluded topological minor was discovered by Grohe and Marx [6] and with an excluded immersion by Wollan [14].

Here we obtain, as a consequence of Theorem 1.3, a structural description of graphs with a forbidden lift-contraction. Informally, for a fixed graph  $H$ , any graph  $G$  that does not contain  $H$  as a lift-contraction contains a set of vertices  $R$  whose size depends only on the excluded graph  $H$  such that every connected component of  $G[V \setminus R]$  is of treewidth at most 2 and has at most two neighbors in  $R$ . A simple corollary of our structural result is that graphs with an excluded lift-contraction are of bounded treewidth and thus of bounded chromatic number.

<sup>1</sup>  $H$  is a topological minor of  $G$ , when some subdivision of  $H$  is a subgraph of  $G$ .

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