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The chromatic number of random Cayley graphs



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ABSTRACT

We consider the typical behavior of the chromatic number of a random Cayley graph of a given group of order n with respect to a randomly chosen set of size $k \leq n/2$. This behavior depends on the group: for some groups it is typically 2 for all $k < 0.99 \log_2 n$, whereas for some other groups it grows whenever k grows. The results obtained include a proof that for any large prime p , and any $1 \leq k \leq 0.99 \log_3 p$, the chromatic number of the Cayley graph of Z_p with respect to a uniform random set of k generators is, asymptotically almost surely, precisely 3. This strengthens a recent result of Czerwiński. On the other hand, for $k > \log p$, the chromatic number is typically at least $\Omega(\sqrt{k/\log p})$ and for $k = \Theta(p)$ it is $\Theta(\frac{p}{\log p})$.

For some non-abelian groups (like $SL_2(Z_q)$), the chromatic number is, asymptotically almost surely, at least $k^{\Omega(1)}$ for every k , whereas for elementary abelian 2-groups of order $n = 2^t$ and any k satisfying $1.001t \leq k \leq 2.999t$ the chromatic number is, asymptotically almost surely, precisely 4. Despite these discrepancies between different groups, it seems plausible to conjecture that for any group of order n and any $k \leq n/2$, the typical chromatic number of the corresponding Cayley graph cannot differ from k by more than a poly-logarithmic factor in n .

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1. Introduction

Let B be a finite group of order n . For an integer $k \leq n/2$, let S be a random subset of B obtained by choosing, randomly, uniformly and independently (with repetitions), k elements of B , and by letting S be the set of these elements and their inverses, without the identity. Thus S is a set of cardinality at

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most $2k$, and is typically of cardinality at least $k - O(k^2/n)$. (Note that some elements b of the group may be of order 2, and if such a b lies in S then $b = b^{-1}$.) In this paper we consider the behavior of the chromatic number of the Cayley graph of B with respect to S , that is, the graph whose vertices are all members of B where b_1 and b_2 are adjacent if $b_1 \cdot b_2^{-1} \in S$. We denote this random graph by (B, k) , and its chromatic number by $\chi(B, k)$.

One motivation for studying this problem are the constructions in [6] in which random self-complementary Cayley graphs of high chromatic number are used in the investigation of a problem in Information Theory, providing graphs with a big gap between their chromatic number and their so-called Witsenhausen rate—see [6] for more details. Another motivation is the fact that many of the known constructions of expanders, like the ones in [5,19,20] are Cayley graphs, the fact that random Cayley graphs with logarithmic degrees over any group are typically expanders [8], and over some groups even a bounded degree suffices [9], and the fact that graphs with strong expansion properties have high chromatic numbers. Yet another reason is the study of an extremal problem of Green regarding sumsets in finite fields, whose investigation leads to the question of estimating the typical behavior of the chromatic number of random Cayley sum graphs of $\mathbb{Z}_p$. See [14,1] for more details.

Our results are asymptotic and we are interested in the case of large n , where k may grow with n or stay constant. As usual, we will say that a property holds asymptotically almost surely (a.a.s., for short), if the probability it holds tends to 1 as n tends to infinity. The problem of determining the typical asymptotic behavior of $\chi(B, k)$ for a general given group B of order n and general $k \leq n/2$ appears to be very difficult, but we do obtain several nontrivial estimates for general groups, as well as more accurate estimates for specific groups.

The rest of the paper is organized as follows. In the next section we consider general groups, cyclic groups are considered in Section 3 and abelian ones in Section 4. The final Section 5 contains several open problems. Throughout the paper all logarithms are in base 2, unless otherwise specified, and we omit floor and ceiling signs whenever these are not crucial. We generally make no serious attempts to optimize the absolute constants in (most of) our estimates.

2. General groups

Note that (B, k) is regular of degree at most $2k$, and hence always $\chi(B, k) \leq 2k + 1$.

Theorem 2.1. *For any group B of order n and any $k \leq n/2$, the chromatic number $\chi(B, k)$ satisfies, a.a.s, the following bounds.*

- (i) $\chi(B, k) \leq O(k/\log k)$.
- (ii) $\chi(B, k) \geq \Omega((k/\log n)^{1/2})$.
- (iii) $\chi(B, k) \geq \Omega(\frac{k^2}{n \log^2 n})$.

Note that the bound (ii) is better than (iii) if $k < cn^{2/3} \log n$ for an appropriate constant c , and that (i) and (iii) imply that for $k = \Theta(n)$,

$$\Omega\left(\frac{n}{\log^2 n}\right) \leq \chi(B, k) \leq O\left(\frac{n}{\log n}\right).$$

In order to prove the theorem we need several lemmas. The first two supply upper bounds for the chromatic number of sparse or pseudo-random graphs.

Lemma 2.2 ([2]). *The chromatic number of any graph with maximum degree d in which every neighborhood of a vertex spans at most d^2/f edges, where $f < d^2$, is $O(d/\log f)$.*

Lemma 2.3 ([3]). *The chromatic number of any d -regular graph with all nontrivial eigenvalues bounded in absolute values by λ is at most*

$$O\left(\frac{d - \lambda}{\log\left(\frac{d - \lambda}{\lambda + 1} + 1\right)}\right).$$

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