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A nilpotent Freiman dimension lemma

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ABSTRACT

We prove that a K-approximate subgroup of an arbitrary torsion-free nilpotent group can be covered by a bounded number of cosets of a nilpotent subgroup of bounded rank, where the bounds are explicit and depend only on K. The result can be seen as a nilpotent analogue to Freiman's dimension lemma.

To the memory of Yahya Ould Hamidoune

1. Introduction

Freiman's theorem (see e.g. [16, Theorem 5.33]) asserts that any finite subset $A \subseteq \mathbb{Z}$ with *doubling* K, that is to say with $|A+A| \leqslant K|A|$ for some parameter $K \geqslant 1$, is contained in a generalised arithmetic progression P with rank $O_K(1)$ and size $O_K(1)|A|$. A number of papers have appeared in the past few years attempting to prove analogues of this result for groups other than the additive group of integers, and in particular for non-commutative groups. See, for example, [1–4,6,7,9–12,14,15].

In [5], we established a structure theorem for sets of small doubling and for *approximate subgroups* of arbitrary groups. Approximate subgroups are symmetric finite subsets A of an ambient group whose product set AA can be covered by a bounded number of translates of the set. In particular they are sets of small doubling. They were introduced in [14], because not only are they easier to handle than arbitrary sets of small doubling, but also the study of arbitrary sets of small doubling reduces to a large extent to that of approximate subgroups. We refer the reader to Section 2 for a precise definition and a reminder of their basic properties.

One of the main theorems proved in [5] asserts, roughly speaking, that approximate subgroups can be covered by a bounded number of cosets of a certain finite-by-nilpotent subgroup with bounded complexity. The bounds on the number of cosets and on the complexity (rank and step) of the nilpotent subgroup depend only on the doubling parameter K.

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It turns out that there is an intimate analogy between approximate groups and neighbourhoods of the identity in locally compact groups. In [5], this analogy was worked out making use of ultrafilters in order to build a certain limit approximate group to which we then applied an adequately modified version of the tools developed in the 1950s for the solution of Hilbert's fifth problem on the structure of locally compact groups.

The use of ultrafilters makes the results of [5] ineffective in a key way. The aim of the present note is to give, for a fairly special case, a simple argument independent of [5] but which achieves the same goal with explicit bounds. Our main result is the following.

Theorem 1.1. *Let G be a simply connected nilpotent Lie group and let A be a K-approximate subgroup of G. Then A can be covered by at most $\exp(K^{O(1)})$ cosets of a closed connected Lie subgroup of dimension at most $K^{O(1)}$.*

The main feature of this theorem is that G can be of arbitrary dimension and nilpotency class. One can be completely explicit about the bounds in this theorem as well as in the corollary below, and take K^{2+29K^9} for the first bound on the number of cosets and K^9 for the bound on the dimension.

Theorem 1.1 can be compared with Freiman's dimension lemma (see [16, Theorem 5.20]), according to which a finite set with doubling at most K in an arbitrary vector space over $\mathbb{R}$ is contained in an affine subspace with dimension bounded at most $\lfloor K-1 \rfloor$.

We note that [5] contains a result similar to Theorem 1.1, and even with the better bound $O(\log K)$ for the dimension of the nilpotent subgroup. Unlike Theorem 1.1, however, this provides no explicit bound whatsoever on the number of translates.

Corollary 1.2. *Let G be a residually (torsion-free nilpotent) group and let A be a finite K-approximate subgroup of G. Then A can be covered by at most $\exp(K^{O(1)})$ cosets of a nilpotent subgroup of nilpotency class at most $K^{O(1)}$.*

Remark. A group G is residually (torsion-free nilpotent) if, for every $g \in G \setminus \{1\}$, there is a homomorphism $\pi : G \to Q$ where Q is torsion-free nilpotent and $\pi(g) \neq \mathrm{id}$. This class of groups includes many non-nilpotent groups, such as all finitely generated free groups.

The main idea behind the proof of Theorem 1.1 and Corollary 1.2 is encapsulated in a simple lemma, Lemma 3.1. This lemma is essentially due to Gleason [8], who uses an analogous idea in order to establish that the class of compact connected subgroups of a given locally compact group admits a maximal element.

The paper is organised as follows. In Section 2, we recall the definition and some basic properties of approximate groups. In Section 3, we state and prove our key lemma. In Section 4, we establish that approximate subgroups of residually nilpotent groups have elements with a large centralizer. Finally in Section 5, we complete the proof of the main results.

2. Basic facts on approximate groups

In this section we recall some basic properties of approximate groups. Background on approximate groups and their basic properties can be found in the third author's paper [14].

Definition 2.1 (*Approximate Groups*)**.** Let $K \geqslant 1$. A finite subset A of an ambient group G is called a *K-approximate subgroup* of G if the following properties hold:

(i) the set A is symmetric in the sense that $\mathrm{id} \in A$ and $a^{-1} \in A$ if $a \in A$;
(ii) there is a symmetric subset $X \subseteq A^3$ with $|X| \leqslant K$ such that $A \cdot A \subseteq X \cdot A$.

We record the following important, yet easily shown, fact.

Lemma 2.2. *Let A be a K-approximate subgroup in an ambient group G and H a subgroup of G. Then we have the following facts.*

(i) $|A| \leqslant |A^2 \cap H||AH/H| \leqslant |A^3|$.
(ii) *$A^2 \cap H$ is a K^3-approximate subgroup of G. Moreover $|A^k \cap H| \leqslant K^{k-1}|A^2 \cap H|$ for all $k \geqslant 1$.*
(iii) *If $\pi : G \to Q$ is a homomorphism then $\pi(A)$ is a K-approximate subgroup of Q.*