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Contractions and expansion

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ABSTRACT

Let $A \subseteq \mathbb{R}$ be a finite set and let $K \geqslant 1$ be a real number. Suppose that for each $a \in A$ we are given an injective map $\phi_a : A \to \mathbb{R}$ which fixes a and contracts other points towards it in the sense that $|a - \phi_a(x)| \leqslant \frac{1}{K}|a - x|$ for all $x \in A$, and such that $\phi_a(x)$ always lies between a and x. Then

$$\left|\bigcup_{a \in A} \phi_a(A)\right| \geqslant \frac{K}{10}|A| - O_K(1).$$

An immediate consequence of this is the estimate $|A + K \cdot A| \geqslant \frac{K}{10}|A| - O_K(1)$, which is a slightly weakened version of a result of Bukh.

To the memory of Yahya Hamidoune

1. Introduction

In this short note we consider the behaviour of a set $A \subseteq \mathbb{R}$ under a collection of maps $\phi_a : A \to \mathbb{R}$. Let $K \geqslant 1$ be a parameter. We will assume that these maps have the following properties:

(i) ϕ_a is injective;
(ii) $\phi_a(a) = a$;
(iii) ϕ_a is a K-contraction in the sense that $|a - \phi_a(x)| \leqslant \frac{1}{K}|a - x|$ for all $x \in A$;
(iv) $\phi_a(x)$ lies between a and x.

Theorem 1. *Suppose that $A \subseteq \mathbb{R}$ is a finite set of size n and that we have maps ϕ_a as above. Then*

$$\left|\bigcup_{a \in A} \phi_a(A)\right| \geqslant \frac{1}{8}Kn(1 - n^{-1/K^2}).$$

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Remark. The bound that we have given here looks a little odd, but it is convenient for our proof. Note that it is at least $\frac{1}{10}Kn - O(e^{CK^2})$, a slightly more precise version of the bound stated in the abstract.

An immediate corollary of this theorem is the following. Here, $A + K \cdot A := \{a + Ka' : a, a' \in A\}$.

Corollary 1. *Suppose that $A \subseteq \mathbb{R}$ is a finite set and that $K \geqslant 1$ is a real number. Then $|A + K \cdot A| \geqslant \frac{1}{10}K|A| - O(e^{CK^2})$.*

Proof. Simply apply the theorem with $\phi_a(x) := (x+Ka)/(K+1)$. These maps obviously verify (i)–(iv) above. $\square$

We note that Bukh [1] established a much more precise result when $K \in \mathbb{Z}$, namely that $|A + K \cdot A| \geqslant (|K| + 1)|A| - o(|A|)$. Assuming that K is an integer should not make things any easier, and furthermore our approach would appear not to generalise to the more general sums of dilates $\lambda_1 \cdot A + \cdots + \lambda_t \cdot A$ considered by Bukh. Let us also note that Cilleruelo, Hamidoune and Serra [2] obtained an extremely precise result when K is prime, establishing that $|A + K \cdot A| \geqslant (K+1)|A| - \lceil K(K+2)/4 \rceil$ for $|A| \geqslant 3(K-1)^2(K-1)!$.

2. Proof of the main theorem

In this section we prove Theorem 1. Let $F(n)$ be the minimum size of $\bigcup_{a\in A} \phi_a(A)$ over all sets A of size n. We will obtain a lower bound for $F(n)$ in terms of the values of $F(n')$, $n' < n$; we may then proceed by induction.

We will use the (obvious) *convexity* property of maps ϕ_a satisfying (i)–(iv) above, namely that $\phi_a(I) \subseteq I$ for any interval I containing a.

We clearly have $F(1) = 1$, so suppose that $A \subseteq \mathbb{R}$ is a set of size $n \geqslant 2$. We may rescale in such a way that the extreme points of A are 0 and 1. Suppose that there is some $a_* \in A$ such that $|A \cap [a_* - 1/K, a_* + 1/K]| \leqslant 6n/K$. Write $A_1 := A \cap [0, a_* - 1/K)$ and $A_2 := A \cap (a_* + 1/K, 1]$, and set $n_1 := |A_1|$, $n_2 := |A_2|$. Then ϕ_{a_*} contracts all of A into the interval $[a_* - 1/K, a_* + 1/K]$. By induction and the convexity property we have

$$\begin{aligned} F(n) &\geqslant \left|\bigcup_{a\in A_1} \phi_a(A_1)\right| + |A| + \left|\bigcup_{a\in A_2} \phi_a(A_2)\right| \\ &\geqslant F(n_1) + n + F(n_2). \end{aligned} \tag{2.1}$$

Note that $n_1 + n_2 \geqslant (1 - 6/K)n$.

Alternatively, suppose there is no such a_*. Obviously $A = A \cap \bigcup I_a$, where $I_a = [a - 1/K, a + 1/K]$. We may pass to disjoint subcollections $\bigcup_{a\in S_1} I_a$ and $\bigcup_{a\in S_2} I_a$ whose union is $\bigcup_{a\in A} I_a$ (cf. [3]). By assumption, $|A \cap I_a| > 6n/K$, and therefore $|S_1|, |S_2| < K/6$. It follows that A is covered by $<K/3$ intervals of length $2/K$, and hence there is some $a^* \in A$, $a^* < 1$, such that A is disjoint from $(a^*, a^* + 1/K]$. Set $A_1 := A \cap [0, a^*]$ and $A_2 := (a^* + 1/K, 1]$, and set $n_1 := |A_1|$, $n_2 := |A_2|$; note that $n_1 + n_2 = n$. Note also that $\phi_{a^*}(A_2) \subseteq (a^*, a^* + 1/K]$; here, we make crucial use of property (iv), which asserts that $\phi_{a_*}(x)$ lies between a_* and x.

By the convexity property and the above observations,

$$F(n) \geqslant \left|\bigcup_{a\in A_1} \phi_a(A_1)\right| + |A_2| + \left|\bigcup_{a\in A_2} \phi_a(A_2)\right| \geqslant |A_2| + F(n_1) + F(n_2).$$

Note, however, that A_2 contains 1 and hence $A \cap I_1$, a set of size $>6n/K$. Therefore

$$F(n) \geqslant \frac{6n}{K} + F(n_1) + F(n_2) \tag{2.2}$$

in this case.