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Cayley digraphs and graphs

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ABSTRACT

We shortly recall some definitions that involve refinements of connectivity, and a theorem of Y.O. Hamidoune. We consider some aspects of Cayley digraphs and vertex- and arc-transitive digraphs that he investigated.

In memoriam, Y.O. Hamidoune

1. Cayley digraphs

We just recall here well-known facts.

Cayley digraphs are defined with a group G and a subset S of G: the vertices of the Cayley digraph $\mathrm{Cay}(G, S)$ are the elements of the group, and its arcs are all the couples (a, as) with $a \in G$ and $s \in S$.

The digraph $\mathrm{Cay}(G, S)$ is strongly connected if and only if S generates G.

To avoid loops, one has to forbid the presence of the unit of G in S.

The digraph is symmetric if $S = S^{-1}$. In this case, it can be considered as a graph.

The Cayley digraph $\mathrm{Cay}(G, S)$ is vertex-transitive, owing to the digraph automorphisms $t_a : g \mapsto ag$ with $a \in G$.

2. Fragments and atoms

Y.O. Hamidoune developed W. Mader's ideas on connectivity of digraphs. A cut-set is a set of vertices whose deletion leaves a digraph that is no longer strongly connected. If the underlying graph is not complete, then of course the in-neighbourhood, or the out-neighbourhood, of any vertex constitutes a cut-set, but not always of minimum order. One is interested in cut-sets of minimal order, this order is the *connectivity* κ of the digraph. A set whose out-neighbourhood is a cut-set T of order κ, other than the complement of T in the digraph, is called a *positive fragment*. One defines similarly the notion of *negative fragment*. A fragment (positive or negative) of minimal order is called an *atom*. By definition, non-complete digraphs always have atoms but they do not necessarily have both positive and negative atoms.

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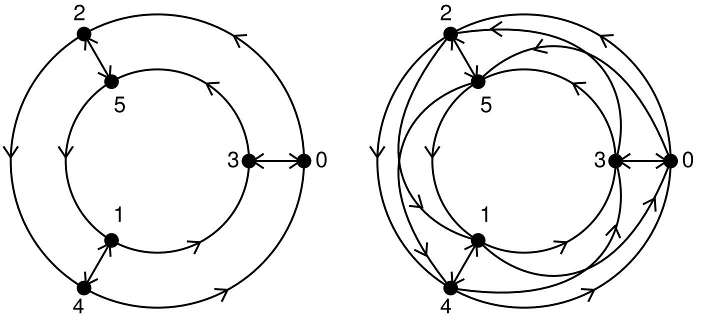

Fig. 1. Two digraphs from $\mathbb{Z}/6\mathbb{Z}$.

W. Mader [3] had already proved that when an atom meets a fragment of a non-directed graph, then the atom is entirely contained in that fragment. Y.O. Hamidoune [1] extended this result: a positive fragment F and a positive atom A satisfy

$$A \cap F = A \quad \text{or} \quad \emptyset$$

and of course the same relation holds for negative fragments and atoms.

Y.O. Hamidoune [1] also proved that the atoms of a Cayley digraph on a group G are a subgroup R of the group, and of course its translates xR, $x \in G$: more precisely, R is the atom containing the identity element and is generated by its elements that belong to S.

2.1. Example

The Cayley digraph on $\mathbb{Z}/6\mathbb{Z}$, with generating set $\{2, 3\}$, shown in Fig. 1 left admits the cut-set $\{2, 3\}$ (of minimum order), that is the out-neighbourhood of 0, or the in-neighbourhood of 5 as well. So the vertices are atoms (positive and negative). But there are other cut-sets with 2 vertices, for example $\{2, 5\}$ is the out-neighbourhood of $\{1, 4\}$ and the in-neighbourhood of $\{0, 3\}$. Hence, besides vertices, we find fragments with 2 vertices. Here the atoms are the vertices.

2.2. Example

The Cayley digraph on $\mathbb{Z}/6\mathbb{Z}$, with generating set $\{2, 3, 5\}$, shown in Fig. 1 right admits the cut-set $\{2, 5\}$ (of minimum order), that is the out-neighbourhood of $\{1, 4\}$ and the in-neighbourhood of $\{0, 3\}$. Any vertex has 3 out-neighbours and 3 in-neighbours. So the atoms here have 2 vertices. They are the translates of the subgroup $\{0, 3\}$.

2.3. An infinite example

The positive and negative fragments may have different sizes, as shown by the infinite tree where each vertex has out-degree 2 and in-degree 1. The connectivity is 1, the positive fragments have one element and the negative ones have two (Fig. 2).

3. Reversing the arcs

For a finite group, it may happen that reversing all arcs gives an isomorphic digraph. The digraph will then be said to be *self-reverse*. The positive and negative fragments then have the same size.

This is always the case for Cayley digraphs on abelian groups. Indeed, reversing the arcs amounts to replacing the generator S by the set S^{-1} of the inverses of the elements of S. If the group G is abelian, the application $g \mapsto g^{-1}$ of the group onto itself is a group isomorphism, proving that $\mathrm{Cay}(G, S)$ is isomorphic to its reverse $\mathrm{Cay}(G, S^{-1})$.