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On the measure of large values of the modulus of a trigonometric sum

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A R T I C L E   I N F O

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A B S T R A C T

We study the connection between the additive structure of a finite set $A \subset \mathbb{Z}$ and the measure of large values of the modulus of a trigonometric sum.

1. Introduction

Let $\mathbb{Z}$ be the ring of integers, $\mathbb{R}$ the field of real numbers, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$,

$$e(\alpha) = e^{2\pi i\alpha}, \quad i = \sqrt{-1}, \qquad [a, b] = \{a, a+1, \dots, b\}, \quad a, b \in \mathbb{Z},\ a < b.$$

For a set $A \subset \mathbb{Z}$, $|A| = \text{card}(A)$ denotes the number of elements of A and $\ell(A) = \max A - \min A$. Denote

$$S_A(\alpha) = \sum_{a\in A} e(\alpha a), \quad \alpha \in \mathbb{T},$$

$$E(A, \lambda|A|) = \{\alpha \in \mathbb{T} : |S_A(\alpha)| \geq \lambda|A|\},$$

and

$$\mu^*_\lambda(m) = \sup\{\text{mes}\, E(A, \lambda m) : A \subset \mathbb{Z},\ |A| = m,\ \ell(A) \leq 2m\}.$$

In this paper, we study the following problem: find the set A for which mes $E(A, \lambda|A|)$ has the maximal value assuming that $|A|$ is given and A lies in a "short" interval.

The problem has a rather long history (see [6,4,9,12]). Connections of this theme with problems of probability theory, harmonic analysis and coding theory are shown in [12,11,8,2]. Results in this

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direction can also be found in [5,1,3]. There exists very strong connections of this field with Roth's three-term arithmetic progression problem [9, pp. 140–142].

In this paper, we shall prove the following.

Theorem 1. *Let $A \subset [-N, N]$, $|A| = k \geq N + 1$, where N is a sufficiently large integer. Then for $\lambda \geq \frac{2\sqrt{2}}{\pi} = 0.90032$,[2] the value $\mu^*(|A|)$ is attained if and only if A is an arithmetic progression.*

Let us stress that the conditions of Theorem 1 imply that the difference of the corresponding arithmetic progression A is equal to 1, and only in the case $k = N + 1$ is equal to 1 or 2.

An example: if $|A| = N + 1$, then

$$\mu^*(N+1) = \text{mes}\left\{\alpha \in \mathbb{T} : \left|\sum_{a=0}^{N} e(\alpha a)\right| \geq \frac{2\sqrt{2}}{\pi}(N+1)\right\} = \frac{2\theta}{N+1} + O\left(\frac{1}{N^3}\right),$$

where θ is a solution of the equation

$$\frac{\sin \pi\theta}{\pi\theta} = \frac{2\sqrt{2}}{\pi},$$

so that

$$\pi\theta = 0.775.$$

Let us review existing results in more detail and comment on some existing and possible applications.

The problem of finding the maximal measure $\mu_{\max} = \sup_A \mu$ for sets of α for which $|S_A(\alpha)|$ is bigger than some given number which is less than the trivial estimate equal to k, and of finding the sets A with this property was first formulated in [9, p. 144].

The case when

$$|S_A(\alpha)| \geq (1 - \varepsilon)|A|$$

and $\varepsilon = o(1)$ was treated by A. A. Yudin in [12].

The case when ε is some positive constant was studied by A. Besser in [2]. However, the constant achieved turned out to be very small $\left(\varepsilon = \frac{1}{20000}\right)$, and attempts at further progress encountered major technical difficulties.

This is why in [5] it was proposed to add an additional condition and to study only those sets included in a segment that is not too long.

Let us now describe a connection with problems of information transfer. The code word $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1})$, where $\varepsilon_i \in \{+1, -1\}$ for all i, is transmitted with the aid of a signal

$$F_n(t) = \sum_{j=0}^{n-1} \varepsilon_i e(jt).$$

Technical conditions ask that the value

$$\max_{t \in \mathbb{T}} |F_n(t)|$$

be as small as possible.

We have

$$F_n(t) = 2 \sum_{k:\varepsilon_k = 1} e(kt) - D_n(t),$$

[2] The result stated here for $\lambda = 0.90032$ is in fact valid for $\lambda = 0.75$. This last result was obtained by means of computations too tedious to be included in this paper.

In general, the choice of the value of λ in the formulation of the theorem (in our case $\lambda = 2\sqrt{2}/\pi \approx 0.9$) is determined by the wish to balance between the strength of the result (smaller values of λ) and the complexity of the computations, which is bigger for smaller values of λ.