European Journal of Combinatorics 34 (2013) 69–81

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Linear time construction of a compressed Gray code

Darko Dimitrov [a], Tomáš Dvořák [b], Petr Gregor [b], Riste Škrekovski [c]

[a] Institut für Informatik, Freie Universität, Berlin, Germany

[b] Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

[c] Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia

ARTICLE INFO

Article history:
Available online 24 August 2012

ABSTRACT

An n-bit (cyclic) Gray code is a (cyclic) ordering of all n-bit strings such that consecutive strings differ in exactly one bit. We construct an n-bit cyclic Gray code $\mathbf{C}_n$ whose graph of transitions is isomorphic to an induced subgraph of the d-dimensional hypercube where $d = \lceil \lg n \rceil$. This allows to represent $\mathbf{C}_n$ so that only $\Theta(\log\log n)$ bits per n-bit string are needed. We provide an explicit description of an algorithm which generates the transition sequence of $\mathbf{C}_n$ in linear time with respect to the output size.

1. Introduction

An n-bit *Gray code* $\mathbf{C}_n = (u_1, u_2, \ldots, u_N)$, $N = 2^n$, is a sequence of all n-bit strings such that consecutive strings differ in exactly one bit. If, moreover, the first and the last strings also differ in exactly one bit, the code is called *cyclic*.

Gray codes are named after Frank Gray, who in 1953 patented a simple scheme to generate such a cyclic code for every $n \geq 1$ [4]. The resulting code $\mathbf{\Gamma}_n$, known as a *binary reflected Gray code*, may be defined recursively by

$$\mathbf{\Gamma_1} = (0, 1), \qquad \mathbf{\Gamma}_{n+1} = 0\mathbf{\Gamma}_n, 1\mathbf{\Gamma}_n^R \tag{1.1}$$

where $b\mathbf{\Gamma}_n$ denotes the sequence $\mathbf{\Gamma}_n$ with $b \in \{0, 1\}$ prefixed to each string, and $\mathbf{\Gamma}_n^R$ denotes $\mathbf{\Gamma}_n$ in reverse order [6]. Gray codes have found applications in such diverse areas as image processing, signal encoding or data compression, and the research of alternative constructions of Gray codes with additional properties has received a good deal of attention [9].

E-mail addresses: Darko.Dimitrov@inf.fu-berlin.de (D. Dimitrov), Tomas.Dvorak@mff.cuni.cz (T. Dvořák), Petr.Gregor@mff.cuni.cz (P. Gregor).

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2012.07.015

$$\mathbf{C}_4 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0, & 0, & 0, & 0, & 0, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 0, & 0, & 0 \end{pmatrix}$$

$$\tau(\mathbf{C}_4) = [0, \ 1, \ 0, \ 2, \ 3, \ 2, \ 0, \ 1, \ 0, \ 2, \ 0, \ 1, \ 3, \ 1, \ 0, \ 2]$$

Fig. 1. The cyclic Gray code $\mathbf{C}_4$ and its transition sequence $\tau(\mathbf{C}_4)$.

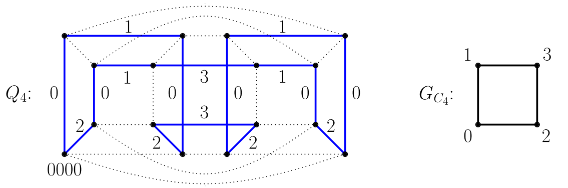

Fig. 2. The cyclic Gray code $\mathbf{C}_4$ depicted as a Hamiltonian cycle of Q_4 and its graph of transitions G_{C_4}.

Let $\mathbf{C}_n$ be an n-bit cyclic Gray code. A *transition* $\Delta(x, y)$ between two consecutive strings $x = x_0x_1 \cdots x_{n-1}$ and $y = y_0y_1 \cdots y_{n-1}$ of $\mathbf{C}_n$ is the unique integer $i \in [n] = \{0, 1, \ldots, n-1\}$ such that $x_i \neq y_i$. A *transition sequence* $\tau(\mathbf{C}_n)$ of $\mathbf{C}_n = (u_1, u_2, \ldots, u_N)$ is the sequence $[t_1, t_2, \ldots, t_N]$ listing transitions $t_i = \Delta(u_i, u_{i+1})$ for all $i \in [N] \setminus \{0\}$ and well as the *closing transition* $t_N = \Delta(u_N, u_1)$. See Fig. 1 for an illustration. Perhaps the most famous transition sequence is known as the "ruler function" [8]:

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1 \ldots.$$

If bits are numbered from the right starting with 1, i.e., $x = x_nx_{n-1} \cdots x_1$ for $x \in \{0, 1\}^n$, each prefix of this sequence of length $2^n - 1$ corresponds to $\tau(\mathbf{\Gamma_n})$ without the closing transition.

A *graph of transitions* of $\mathbf{C}_n$, denoted by G_{C_n}, also called a *graph induced by* $\tau(\mathbf{C}_n)$, is an undirected graph with vertices $V(G_{C_n}) = [n]$ and edges

$$E(G_{C_n}) = \{t_it_{i+1} \mid i \in [N] \setminus \{0\}\} \cup \{t_Nt_1\}.$$

For example, the graph of transitions of the code $\mathbf{C}_4$ from Fig. 1 is a 4-cycle; see Fig. 2. The graph of transitions of $\mathbf{\Gamma}_n$ is star centered at vertex $n-1$ with leaves $[n-1]$, since here $\Delta(u_i, u_{i+1}) = n-1$ iff i is an odd number.

Slater [10] was probably the first who asked which graphs are graphs of transitions of (cyclic) Gray codes. Bultena and Ruskey [1] used computer search to catalog these graphs for $n \leq 5$, and Wilmer and Ernst [13] extended the list to all $n \leq 7$. For larger values of n, there are only some partial results [1,10,12,13]. For example, Bultena and Ruskey [1] proved that every tree of diameter 4 is a graph of transitions of a cyclic Gray code, but no tree of diameter 3 has this property. Wilmer and Ernst [13] showed that for an arbitrarily large $d \geq 4$ there is a Gray code whose graph of transitions is a tree of diameter d. Suparta and van Zanten [12] proved that the complete graphs are also graphs of transitions of cyclic Gray codes. Among many open problems posed in [1,10–13], it is particularly interesting whether paths and cycles are graphs of transitions of (cyclic) Gray codes.

Each n-bit (cyclic) Gray code may be viewed as a Hamiltonian path (cycle) in the n-dimensional hypercube Q_n [3]; see Fig. 2 for an illustration. It is therefore natural to ask whether there are Gray codes whose graphs of transitions are hypercubes. This question has been answered in [2], where we showed that for every positive integer n there exists an n-bit cyclic Gray code $\mathbf{C}_n$ whose graph of transitions is isomorphic to Q_d if $n = 2^d$, or to a subgraph of Q_n if $2^{d-1} < n < 2^d$. In the latter case, the subgraph of Q_d need not be necessarily an induced subgraph.

In this paper, we provide an alternative construction which coincides with that of [2] for $n = 2^d$, but differs from it for $2^{d-1} < n < 2^d$. The resulting n-bit cyclic Gray code then has the property that its graph of transitions is isomorphic to an induced subgraph of Q_d, which has the maximum number