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Enumeration of hybrid domino–lozenge tilings

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ABSTRACT

We solve and generalize an open problem posted by James Propp (Problem 16 in *New Perspectives in Geometric Combinatorics*, Cambridge University Press, 1999) on the number of tilings of quasi-hexagonal regions on the square lattice with every third diagonal drawn in. We also obtain a generalization of Douglas' theorem on the number of tilings of a family of regions of the square lattice with every second diagonal drawn in.

1. Introduction

The field of exact enumeration of tilings (equivalently, perfect matchings) dates back to the early 1900s when MacMahon proved his classical theorem on the number of plane partitions that fit in a given box (see [10]). This theorem is equivalent to the fact that the number of unit rhombus (or lozenge) tilings of a hexagon $H_{a,b,c}$ of side-lengths a, b, c, a, b, c (in cyclic order) drawn on the triangular lattice is equal to

$$\mathrm{M}(H_{a,b,c}) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2} \tag{1.1}$$

(we use the operator M to denote both the number of tilings of a lattice region and the number of perfect matchings of a graph, as the two objects can be identified by a well-known bijection).

Another classical result, from the early 1960s, is the enumeration of domino tilings of a rectangle on the square lattice, due independently to Kasteleyn [9] and Temperley and Fisher [12]. This states that the number of domino tilings of a $2m$ by $2n$ rectangle on the square lattice equals

$$\mathrm{M}(G_{2m,2n}) = 2^{2mn}\prod_{j=1}^{m}\prod_{k=1}^{n}\left(\cos^2\left(\frac{j\pi}{2m+1}\right)+\cos^2\left(\frac{k\pi}{2n+1}\right)\right). \tag{1.2}$$

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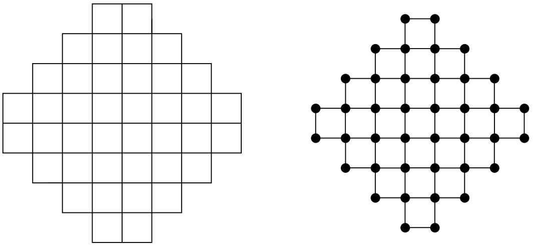

Fig. 1.1. The Aztec diamond region (left) and the Aztec diamond graph (right) of order 4.

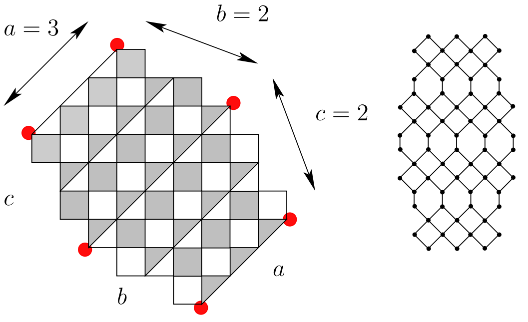

Fig. 2.1. An example of the quasi-hexagonal regions considered in [11], Problem 16 (left), and its dual graph (right).

In the early 1990s, Elkies, Kuperberg, Larsen and Propp [7] considered another family of simple regions on the square lattice called Aztec diamonds (see Fig. 1.1 for an example), and proved that the number of domino tilings of the Aztec diamond of order n is given by the simple formula

$$\mathrm{M}(AD_n) = 2^{n(n+1)/2}. \tag{1.3}$$

A large body of related work followed (see e.g. [2,3,5,6,8,13], and the references in [11] for a more extensive list), centered on families of lattice regions whose tilings are enumerated by simple product formulas. The state of affairs at the end of that decade is captured by Propp's paper [11] published in 1999, which presented a list of 32 open problems in the field of enumeration of perfect matchings.

Most of those 32 problems have been solved in the meanwhile, but some are still open. In this paper we solve and generalize one of these open problems (Problem 16 on Propp's list [11]). Our methods also provide a new proof and a generalization for a related result of Douglas [6].

2. Statement of main results

Problem 16 on Propp's list [11] concerns a family of quasi-hexagonal regions on the lattice obtained from the square lattice by drawing in every third southwest-to-northeast diagonal. The case when the side-lengths of the quasi-hexagon are 3, 2, 2, 3, 2, 2 (clockwise from top) is illustrated in Fig. 2.1. Problem 16 of [11] asks for a formula for the number of tilings of the quasi-hexagon of sides a, b, c, a, b, c (the sides of length a are the ones along *diagonals*[1] of the square lattice), where the allowed tiles are unions of two fundamental regions of the resulting dissection of the square lattice sharing an edge.

As mentioned in [11], the case $a = b = c$ (to which also the cases $a = b < c$ and, by symmetry, $a = c < b$ turn out to reduce) has been solved by Ben Wieland (in unpublished work), who showed that the number of tilings is given in these situations by powers of 2.

[1] From now on, “diagonal(s)” refers to “southwest-to-northeast diagonal(s)”.