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Enumerating colorings, tensions and flows in cell complexes

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ABSTRACT

We study quasipolynomials enumerating proper colorings, nowhere-zero tensions, and nowhere-zero flows in an arbitrary CW-complex X, generalizing the chromatic, tension and flow polynomials of a graph. Our colorings, tensions and flows may be either *modular* (with values in $\mathbb{Z}/k\mathbb{Z}$ for some k) or *integral* (with values in $\{-k+1,\dots,k-1\}$). We obtain deletion–contraction recurrences and closed formulas for the chromatic, tension and flow quasipolynomials, assuming certain unimodularity conditions. We use geometric methods, specifically Ehrhart theory and inside-out polytopes, to obtain reciprocity theorems for all of the aforementioned quasipolynomials, giving combinatorial interpretations of their values at negative integers as well as formulas for the numbers of acyclic and totally cyclic orientations of X.

1. Introduction

This article is about generalizing the enumeration of colorings, flows, cuts and tensions from graphs to cell complexes. We begin with a review of colorings of graphs.

Let $G=(V,E)$ be a graph and let k be a positive integer. A *proper k-coloring* of G is a function $f:V\to K$, where K is a "palette" of size k and $f(v)\neq f(w)$ whenever vw is an edge of G. It is well known that the number $\chi_G(k)$ of proper k-colorings is a polynomial in k (the *chromatic polynomial*

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of G). Remarkably, the numbers $\chi_G(-k)$ have combinatorial interpretations as well, as discovered by Stanley [33]; the best known of these is that $|\chi_G(-1)|$ is the number of acyclic orientations of G. This phenomenon is an instance of *combinatorial reciprocity* and is closely related to, e.g., Ehrhart's enumeration of lattice points in polytopes [19] and Zaslavsky's theorems on counting regions of hyperplane arrangements [38].

Fix an orientation on the edges of G and let ∂ be the signed incidence matrix of G; that is, the rows and columns of ∂ are indexed by vertices and edges respectively, and the (v, e) entry is

$$\begin{cases} 1 & \text{if vertex } v \text{ is the head of edge } e, \\ -1 & \text{if vertex } v \text{ is the tail of edge } e, \\ 0 & \text{if } v \text{ is not an endpoint of } e. \end{cases}$$

If we regard a coloring f as a row vector $c = [f(v)]_{v\in V}$, then properness says precisely that $c\partial$ is nowhere zero, since for each edge vw, the corresponding entry of $c\partial$ is $f(v) - f(w)$. Here the palette may be regarded either as the integers $1, \dots, k$ or as the elements of $\mathbb{Z}/k\mathbb{Z} = \mathbb{Z}_k$. The number of k-colorings may then be computed by linear algebra and inclusion–exclusion, yielding Whitney's formula for the chromatic polynomial (see, e.g., [37, §5.3]).

The *cut* and *flow spaces* of G are respectively the row space and kernel of its boundary matrix ∂, regarded as a map of modules over a ring A (typically either $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{Z}_k$). Cuts and flows arise in algebraic graph theory and are connected to the critical group and the chip-firing game; see, e.g., [1] or [21, Chapter 14]. Over $\mathbb{Z}$, we consider in addition the space of *tensions*, or integer vectors of whom some multiple is in the cut space. (Over a field or a finite ring, all tensions are cuts.) For a generalization of cuts and flows to cell complexes, see [18].

If we regard the graph G as a one-dimensional simplicial complex, the matrix ∂ is just the boundary map from 1-chains to 0-chains. Accordingly, we can replace the graph by an arbitrary d-dimensional CW-complex X and define cellular colorings, flows, tensions and cuts in terms of its top cellular boundary map $\partial : C_d(X;\mathbb{Z}) \to C_{d-1}(X;\mathbb{Z})$.

In Section 3, we study the functions

$$\begin{aligned} \chi^*_X(k) &= \text{number of proper } \mathbb{Z}_k\text{-colorings of } X, \\ \tau^*_X(k) &= \text{number of nowhere-zero } \mathbb{Z}_k\text{-tensions of } X, \\ \varphi^*_X(k) &= \text{number of nowhere-zero } \mathbb{Z}_k\text{-flows of } X. \end{aligned}$$

When X is a graph, these are all polynomials in k (for the tension and flow polynomials, see [36]); in fact, they are specializations of the Tutte polynomial of X. For arbitrary cell complexes, we show that the modular counting functions are always quasipolynomials, and find sufficient conditions on X for them to be genuine polynomials (building on the work of the first author and Y. Kemper [2]). We describe two avenues toward such results, using colorings as an example (the other arguments are similar). First, if ∂ contains a unit entry, then pivoting the matrix ∂ there corresponds to the topological operations of deleting a facet or deformation-retracting it onto a neighboring ridge, and gives rise to a deletion–contraction recurrence for modular colorings. Provided that this "shrinking" process can be iterated, it follows by induction that $\chi^*_X(k)$ is a polynomial. The second approach is a linear-algebraic inclusion–exclusion argument, which produces a closed quasipolynomial formula for $\chi^*_X(k)$ (Theorem 3.4), which is easily seen to be polynomial if every column-selected submatrix of ∂ has no nontrivial invariant factors (a weaker condition than total unimodularity).

In Section 4, we study the integral counting functions

$$\begin{aligned} \chi_X(k) &= \text{number of proper } K\text{-colorings of } X, \\ \tau_X(k) &= \text{number of nowhere-zero } K\text{-tensions of } X, \\ \varphi_X(k) &= \text{number of nowhere-zero } K\text{-flows of } X, \end{aligned}$$

where K is the palette $\{-k+1, -k+2, \dots, k-1\} \subseteq \mathbb{Z}$. When X is a graph, these are all polynomials in k as was shown by Kochol [25,26]. These can be regarded as Ehrhart functions, enumerating lattice