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Increasing forests and quadrangulations via a bijective approach ☆

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In this work, we expose four bijections each allowing to increase (or decrease) one parameter in either uniform random forests with a fixed number of edges and trees, or quadrangulations with a boundary having a fixed number of faces and a fixed boundary length. In particular, this gives a way to sample a uniform quadrangulation with $n+1$ faces from a uniform quadrangulation with n faces or a uniform forest with $n+1$ edges and p trees from a uniform forest with n edges and p trees.

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1. Introduction

Maps are known to have a lot of applications in different fields of mathematics, computer science and physics. These applications strongly rely on their combinatorial structures, which have been widely investigated during the last few decades. A particularly interesting class of maps is the class of quadrangulations with or without a boundary, which are for example natural candidates for discretizing surfaces.

Our interest in this paper, inspired from Rémy's algorithm [7] on growing trees, is in finding a natural way to grow a planar quadrangulation that preserves the uniform measure. For instance, we will present a bijection between quadrangulations with a boundary having n faces and $2p$ half-edges on the boundary carrying some distinguished elements and quadrangulations with a boundary having $n+1$ faces and $2p$ half-edges on the boundary also carrying distinguished elements. Our bijection is designed in such a way that the number of possibilities for distinguishing the necessary elements only depends on the size and boundary length of the quadrangulations. Forgetting these distinguished elements, we obtain a way to sample a uniform quadrangulation with a boundary having a prescribed number of faces and boundary length from a uniform quadrangulation with a boundary having one less face. In other words, there exist some integer constants $c_{n,p}$ and $c'_{n,p}$ such that our construction provides a $c_{n,p}$-to-$c'_{n,p}$ mapping between the set of quadrangulations with a boundary having n faces and $2p$ half-edges on the boundary and the set of quadrangulations with a boundary having $n+1$ faces and $2p$ half-edges on the boundary.

In addition to the probabilistic point of view, such bijections also present a combinatorial interest as they provide an interpretation to some combinatorial identities. The bijections we present in this work interpret already known identities so that they actually provide alternate proofs for these identities. In the future, we hope that other similar bijections will allow to solve some open enumeration problems.

Our method of "cut and glue" bijections, which could informally be pictured as unbuttoning a shirt and buttoning it back incorrectly by putting each button into the hole that immediately follows the correct one, possesses a certain robustness and can be declined in many ways. We present here in detail four such bijections by focusing on forests and quadrangulations with a boundary. In an upcoming work, we plan to present more bijections relying on the same idea. In particular, one of these bijections will allow to recover Tutte's formula [9] counting the number of planar maps with n faces having prescribed degrees $a_1, \dots, a_n$ where at most two a_i's are odd numbers.

It has also been pointed to us that our method somehow recalls a work by Cori [5] where he used a so-called transfer bijection roughly consisting in transferring one degree from a face to a neighboring face. Using a properly defined chain, this allows to transfer one degree from a face to any other face, step by step. We do not believe that his results are related to the present work but we think they are worth mentioning at this point.

2. Setting and presentation of the results

Recall that a planar map is an embedding of a finite connected graph (possibly with loops and multiple edges) into the two-dimensional sphere, considered up to direct homeomorphisms of the sphere. The faces of the map are the connected components of the complement of edges. We will call *half-edge* an edge carrying one of its two possible orientations. We say that a half-edge h is *incident* to a face f (or that f is incident to h) if h belongs to the boundary of f and is oriented in such a way