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An upper bound for nonnegative rank

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ABSTRACT

We provide a nontrivial upper bound for the nonnegative rank of rank-three matrices which allows us to prove that $\lceil 6n/7 \rceil$ linear inequalities suffice to describe a convex n-gon up to a linear projection.

1. Preliminaries

Consider a convex polytope $P \subset \mathbb{R}^n$. An *extension* [5,6] of P is a polytope $Q \subset \mathbb{R}^d$ such that P can be obtained from Q as an image under a linear projection from $\mathbb{R}^d$ to $\mathbb{R}^n$. An *extended formulation* [6,10] of P is a description of Q by linear equations and linear inequalities (together with the projection). The *size* [6,10] of the extended formulation is the number of facets of Q. The *extension complexity* [6,10] of a polytope P is the smallest size of any extended formulation of P, that is, the minimal possible number of inequalities in the description of Q. The number of facets of Q can sometimes be significantly smaller [5] than that of P, and this phenomenon can be used to reduce the complexity of linear programming problems useful for numerous applications [3,5,10].

An important result providing the linear algebraic characterization of extended formulations has been obtained in 1991 by Yannakakis [13]. Let a polytope P (with v vertices and f facets) be defined as the set of all points $x \in \mathbb{R}^n$ satisfying the conditions $c_i(x) \geqslant \beta_i$ and $c_j(x) = \beta_j$, for $i \in \{1, \dots, f\}$ and $j \in \{f+1, \dots, q\}$, where $c_1, \dots, c_q$ are linear functionals on $\mathbb{R}^n$. A slack matrix $S = S(P)$ of P is an f-by-v matrix satisfying $S_{it} = c_i(p_t) - \beta_i$, where $p_1, \dots, p_v$ denote the vertices of P, and we note that S is nonnegative. The following well-known result (see [6, Corollary 5] and also [8, Lemma 3.1]) characterizes the rank of $S(P)$ in terms of the dimension of P.

Proposition 1.1. *A slack matrix of a polytope P has classical rank one greater than the dimension of P.*

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The result by Yannakakis points out the connection between extension complexity and nonnegative factorizations and can now be formulated as follows [6,10,13].

Theorem 1.2. *The extension complexity of a polytope P is equal to the minimal k for which $S(P)$ can be written as a product of f-by-k and k-by-v nonnegative matrices.*

The smallest integer k for which there exists a factorization $A = BC$ with $B \in \mathbb{R}_+^{n\times k}$ and $C \in \mathbb{R}_+^{k\times m}$ is called the *nonnegative rank* of a nonnegative matrix $A \in \mathbb{R}_+^{n\times m}$. Nonnegative factorizations are widely studied and used in data analysis, statistics, computational biology, clustering and numerous other applications [2]. There are still many open questions on nonnegative rank that are interesting for different applications, and a considerable number of them is related to providing bounds on the nonnegative rank in terms of other matrix invariants [4,6,10].

It is easy to show that the nonnegative rank of a matrix equals the classical rank if one of them is less than three [2]. However, even for rank-three m-by-n matrices, the only known upper bound is $\min\{m, n\}$ which is trivial.

Problem 1.3. (See [1].) Assume $n \geqslant 3$. Does there exist a rank-three n-by-n nonnegative matrix with nonnegative rank equal to n?

In view of Proposition 1.1 and Theorem 1.2, one can ask a related question: Does there exist a convex n-gon with extension complexity equal to n, for every n? For $n \leqslant 5$, Problem 1.3 has been solved in the affirmative in [6]. In [7] it was noted that a sufficiently irregular convex hexagon has full extension complexity, providing an affirmative answer for $n = 6$. For $n \geqslant 7$, the problem has been open.

Lin and Chu [11] claimed a positive resolution for Problem 1.3, but their argument has been shown to contain a gap [6,9]. A negative answer for Problem 1.3 has been obtained in [6] for a special case of so-called Euclidean distance matrices. The factorizations of those matrices have been studied subsequently in [9], and the logarithmic upper bounds have been obtained in a number of important special cases. A detailed investigation of extended formulations of convex polygons has been undertaken in [5], but the question about an n-gon with extension complexity equal to n has also been left open.

In our paper we solve Problem 1.3 and prove that for $n > 6$, the answer is negative. In fact, we provide a nontrivial upper bound for the nonnegative rank and prove that an m-by-n rank-three matrix cannot have nonnegative rank greater than $\lceil 6\min\{m, n\}/7\rceil$. From our results it follows that a convex n-gon has extension complexity at most $\lceil 6n/7\rceil$. That is, we prove that any convex n-gon admits a description with $\lceil 6n/7\rceil$ linear inequalities up to a projection.

The organization of the paper is as follows. In Section 2, we prove the main result in a special case of slack matrices of convex heptagons, thus showing that any convex heptagon admits a description with six linear inequalities. In Section 3, we use those results and prove the main results of our paper, which include the upper bound for the extension complexity of a polygon and for the nonnegative rank of a rank-three matrix.

2. Factoring a slack matrix of a convex heptagon

The problem of constructing nonnegative factorizations is rather hard from the computational point of view. Being NP-hard in general [12], this problem can be also difficult to solve even for explicitly written matrices of relatively small size. In fact, the problem of computing the nonnegative ranks of certain n-by-n rank-three matrices with algebraically independent entries remained open for $n = 7$, see [5].

In this section we present a technique that will allow us to factor the matrices of a certain special form, and we will then be able to prove that slack matrices of convex heptagons have nonnegative ranks less than 7. The considerations of this section deal with matrices having not more than seven rows and seven columns, and we adopt the following convention in order to make the presentation more concise.