Journal of Combinatorial Theory, Series B 114 (2015) 70–96

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Minimum vertex degree threshold for loose Hamilton cycles in 3-uniform hypergraphs ☆

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A R T I C L E I N F O

Article history:
Received 14 July 2013
Available online 8 April 2015

Keywords:
Hamilton cycle
Hypergraph
Absorbing method
Regularity lemma

A B S T R A C T

We show that for sufficiently large n, every 3-uniform hypergraph on n vertices with minimum vertex degree at least $\binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c$, where $c = 2$ if $n \in 4\mathbb{N}$ and $c = 1$ if $n \in 2\mathbb{N} \setminus 4\mathbb{N}$, contains a loose Hamilton cycle. This degree condition is best possible and improves on the work of Buß, Hàn and Schacht who proved the corresponding asymptotical result.

1. Introduction

The study of Hamilton cycles is an important topic in graph theory. In recent years, researchers have worked on extending the classical theorem of Dirac [7] on Hamilton cycles to hypergraphs – see recent surveys of [23,20].

Given $k \geq 2$, a k-uniform hypergraph (in short, *k-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k-element subset of V. For $1 \leq l < k$, a k-graph is called an *l-cycle* if its vertices can be ordered cyclically such that each

☆ The second author is partially supported by NSA grant H98230-12-1-0283 and NSF grant DMS-1400073.
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http://dx.doi.org/10.1016/j.jctb.2015.03.007

of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly l vertices. (If we allow $l = 0$, then a 0-cycle is merely a matching and perfect matchings have been intensively studied recently, e.g. [1,5,9,16,15,21,27,30,31].) In k-graphs, a $(k-1)$-cycle is often called a *tight* cycle while a 1-cycle is often called a *loose* cycle. We say that a k-graph contains a *Hamilton l-cycle* if it contains an l-cycle as a spanning subhypergraph. Note that a Hamilton l-cycle of a k-graph on n vertices contains exactly $n/(k-l)$ edges, implying that $k-l$ divides n.

Given a k-graph $\mathcal{H}$ with a set S of d vertices (where $1 \le d \le k-1$) we define $\deg_{\mathcal{H}}(S)$ to be the number of edges containing S (the subscript $\mathcal{H}$ is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(\mathcal{H})$ of $\mathcal{H}$ is the minimum of $\deg_{\mathcal{H}}(S)$ over all d-vertex sets S in $\mathcal{H}$. We refer to $\delta_1(\mathcal{H})$ as the *minimum vertex degree* and $\delta_{k-1}(\mathcal{H})$ the *minimum codegree* of $\mathcal{H}$.

1.1. Hamilton cycles in hypergraphs

Confirming a conjecture of Katona and Kierstead [13], Rödl, Ruciński and Szemerédi [25,26] showed that for any fixed k, every k-graph $\mathcal{H}$ on n vertices with $\delta_{k-1}(\mathcal{H}) \ge n/2 + o(n)$ contains a tight Hamilton cycle. This is best possible up to the $o(n)$ term. With long and involved arguments, the same authors [28] improved this to an exact result for $k = 3$. Loose Hamilton cycles were first studied by Kühn and Osthus [18], who proved that every 3-graph on n vertices with $\delta_2(\mathcal{H}) \ge n/4 + o(n)$ contains a loose Hamilton cycle. Czygrinow and Molla [6] recently improved this to an exact result. The result of Kühn and Osthus [18] was generalized for arbitrary k and arbitrary $l < k/2$ by Hàn and Schacht [10], and independently by Keevash et al. [14] for $l = 1$ and arbitrary k. Later Kühn, Mycroft, and Osthus [17] obtained an asymptotically sharp bound on codegree for Hamilton l-cycles for all $l < k$. Hence, the problem of finding Hamilton l-cycles in k-graphs with large codegree is asymptotically solved.

Much less is known under other degree conditions. Recently Rödl and Ruciński [24] gave a sufficient vertex degree condition that guarantees a tight Hamilton cycle in 3-graphs. Glebov, Person and Weps [8] gave a nontrivial vertex degree condition for tight Hamilton cycles in k-graphs for general k. Neither of these results is best possible – see more discussion in Section 4.

Recently Buß, Hàn, and Schacht [2] studied the minimum vertex degree that guarantees a loose Hamilton cycle in 3-graphs and obtained the following result.

Theorem 1.1. *(See [2, Theorem 3].) For all $\gamma > 0$ there exists an integer n_0 such that the following holds. Suppose $\mathcal{H}$ is a 3-graph on $n > n_0$ with $n \in 2\mathbb{N}$ and*

$$\delta_1(\mathcal{H}) > \left(\frac{7}{16} + \gamma\right)\binom{n}{2}.$$

Then $\mathcal{H}$ contains a loose Hamilton cycle.