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EH-suprema of tournaments with no nontrivial homogeneous sets

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ABSTRACT

A celebrated unresolved conjecture of Erdös and Hajnal states that for every undirected graph H there exists $\epsilon(H) > 0$ such that every undirected graph on n vertices that does not contain H as an induced subgraph contains a clique or stable set of size at least $n^{\epsilon(H)}$.
The conjecture has directed equivalent version stating that for every tournament H there exists $\epsilon(H) > 0$ such that every H-free n-vertex tournament T contains a transitive subtournament of order at least $n^{\epsilon(H)}$. For a fixed tournament H, define $\xi(H)$ to be the supremum of all ϵ for which the following holds: for some n_0 and every $n > n_0$ every tournament with $n \geq n_0$ vertices not containing H as a subtournament has a transitive subtournament of size at least n^{ϵ}. We call $\xi(H)$ the *EH-supremum of H*. The Erdös–Hajnal conjecture is true if and only if $\xi(H) > 0$ for every H. If the conjecture is false then the smallest counterexample has no nontrivial so-called *homogeneous* sets (to be defined below). Therefore of interest are EH-suprema of those tournaments. In [5] it was proven that there exists a constant $\eta > 0$ such that $\xi(H) \leq \frac{4}{h}(1+\eta\frac{\sqrt{\log(h)}}{\sqrt{h}})$ for almost every h-vertex tournament H. However this result does not say anything about $\xi(H)$ for an arbitrarily chosen tournament with no nontrivial homogeneous sets. We address that problem in this paper, proving that there exists $C > 0$ such that every h-vertex tournament H with no nontrivial homogeneous sets satisfies $\xi(H) \leq C\frac{\log(h)}{h}$. We will also give upper bounds on sizes of families of h-vertex tournaments with big EH-suprema. In [1] Alon, Pach and Solymosi proposed a

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procedure that produces bigger graphs satisfying the conjecture from smaller ones. All graphs obtained in such a way have nontrivial homogeneous sets. For a long time that was the only method to obtain infinite families of graphs satisfying the conjecture. Recently Berger, the author and Chudnovsky (see [2]) constructed a new infinite family of tournaments (so-called *galaxies*, to be defined below) that satisfies the conjecture and with no nontrivial homogeneous sets. Therefore it cannot be obtained by the procedure described in [1]. In this paper we construct a new infinite family of tournaments satisfying the conjecture – the family of so-called *constellations* (to be defined below). These results extend the results of [2] since every galaxy is a constellation.

1. Introduction

We denote by $|S|$ the size of a set S. Let G be a graph. We denote by $V(G)$ the set of its vertices. Sometimes instead of writing $|V(G)|$ we will use the shorter notation $|G|$. We call $|G|$ the *size of* G. A *clique* in the undirected graph is a set of pairwise adjacent vertices and a *stable set* in the undirected graph is a set of pairwise nonadjacent vertices. A *tournament* is a directed graph such that for every pair v and w of vertices, exactly one of the edges (v, w) or (w, v) exists. If (v, w) is an edge of the tournament then we say that v is *adjacent to* w and w is *adjacent from* v. For two sets of vertices V_1, V_2 we say that V_1 is *complete to* V_2 (or equivalently V_2 is *complete from* V_1) if every vertex of V_1 is adjacent to every vertex of V_2. A tournament is *transitive* if it contains no directed cycle. For the set of vertices $V = \{v_1, v_2, \ldots, v_k\}$ we say that an ordering $(v_1, v_2, \ldots, v_k)$ is *transitive* if v_1 is adjacent to all other vertices of V, v_2 is adjacent to all other vertices of V but v_1, etc. We denote by $E(G)$ the set of edges of a graph G.

If a tournament T does not contain some other tournament H as a subtournament then we say that T is *H-free*. All logarithms used in the paper are natural logarithms.

A celebrated unresolved conjecture of Erdös and Hajnal is as follows:

1.1. *For any undirected graph H there exists $\epsilon(H) > 0$ such that every n-vertex undirected graph that does not contain H as an induced subgraph contains a clique or a stable of size at least $n^{\epsilon(H)}$.*

In 2001 Alon, Pach and Solymosi proved (see [1]) that Conjecture 1.1 has an equivalent directed version, where undirected graphs are replaced by tournaments and cliques and stable sets by transitive subtournaments.

The equivalent directed version (see [1]) states that:

1.2. *For any tournament H there exists $\epsilon(H) > 0$ such that every n-vertex H-free tournament contains a transitive subtournament of size at least $n^{\epsilon(H)}$.*