Topology and its Applications 196 (2015) 325–346

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Canonical polygons for the hyperbolic structures on the torus with a single cone point ☆

Hirotaka Akiyoshi

Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

ARTICLE INFO

Article history:
Received 26 December 2013
Received in revised form 3 March 2014
Accepted 3 March 2014
Available online 23 May 2015

MSC:
51M10
57M50

Keywords:
Jorgensen theory
Cone hyperbolic structure
Fundamental polyhedron

ABSTRACT

Let T_0 be the once-punctured torus and θ a real number with $0 < \theta < 2\pi$. Let $\widetilde{\rho}$ be a representation of $\pi_1(T_0)$ to $SL(2,\mathbb{R})$ which sends the peripheral loop to an elliptic element with trace $-2\cos(\theta/2)$. Let ρ be the $PSL(2,\mathbb{R})$-representation induced from $\widetilde{\rho}$, and assume it satisfies Bowditch's Q-condition. In this paper, we construct a certain polyhedron, which is obtained as a variation of Jorgensen's theory to cone manifolds, and construct a complete cone hyperbolic structure on the 3-dimensional cone manifold obtained as the product of the torus with a single cone point and $\mathbb{R}$ which induces ρ as the holonomy representation.

1. Introduction

In the famous unfinished manuscript [4], Jorgensen characterized the combinatorial structures of the Ford domains for quasifuchsian punctured torus groups. The characterization seems to work even if we replace the "puncture" with an elliptic point of finite order (as mentioned in [7]). In fact, Jorgensen [5] constructed such an example, and constructed a hyperbolic structure on a closed surface bundle over S^1; to be precise, the structure constructed there is "doubly degenerate" and hence it should be thought of as a limit of quasifuchsian structures. This paper is devoted to the study in an early stage for the project to establish a variant of Jorgensen's theory for the torus with a single cone point.

Let T be the torus and v a point in T, and set $\Sigma = \{v\} \times \mathbb{R} \subset T \times \mathbb{R}$. Let θ be a real number with $0 < \theta < 2\pi$. We denote the triplet $(T, \{v\}, \theta)$ (resp. $(T \times \mathbb{R}, \Sigma, \theta)$) by T_θ (resp. M_θ). Set $T_0 = T - \{v\}$ and $M_0 = T \times \mathbb{R} - \Sigma$. A *cone hyperbolic structure* on M_θ is a metric which is locally isometric to the hyperbolic

☆ This work was supported by JSPS KAKENHI Grant Number 23740064.
E-mail address: akiyoshi@sci.osaka-cu.ac.jp.

http://dx.doi.org/10.1016/j.topol.2015.05.009
0166-8641/

space on M_0, and each point in Σ has a neighborhood which is isometric to a *standard ball of angle* θ with center corresponding to the point. (See Subsection 2.2 for the definition of standard balls *of angle* θ. We refer to [2, Definition 3.9] for a general definition of standard balls.) Then the projection to the first factor induces an isomorphism from $\pi_1(M_0)$ to $\pi_1(T_0)$. We denote the group by G, and fix a pair of generators A_0 and B_0 represented by simple closed curves on T_0 intersecting transversely at a point. Associated with a cone hyperbolic structure on M_θ, we have a smooth hyperbolic structure on M_0, and hence can define the *holonomy representation* $\rho : G \to PSL(2,\mathbb{C})$. Then ρ maps the commutator $[A_0, B_0]$ to a θ-rotation, namely, $\operatorname{tr}\rho([A_0, B_0]) = \pm 2\cos(\theta/2)$. Let $\widetilde{\mathcal{R}}_\theta$ be the $SL(2,\mathbb{C})$-representation space of G consisting of the elements $\widetilde{\rho}$ satisfying $\operatorname{tr}\widetilde{\rho}([A_0, B_0]) = -2\cos(\theta/2)$, and $\mathcal{R}_\theta$ the $PSL(2,\mathbb{C})$-representation space of G obtained from the elements of $\widetilde{\mathcal{R}}_\theta$ by post-composing the canonical surjection $SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$. In this paper, we study the *real slice* $\mathcal{R}_\theta^{\mathbb{R}}$ of $\mathcal{R}_\theta$. To be more specific, let $\widetilde{\mathcal{R}}_\theta^{\mathbb{R}}$ be the subset of $\widetilde{\mathcal{R}}_\theta$ consisting of the conjugacy classes of representations into $SL(2,\mathbb{R})$, and $\mathcal{R}_\theta^{\mathbb{R}}$ the corresponding subset of $\mathcal{R}_\theta$.

The character variety for $\widetilde{\mathcal{R}}_\theta$ is studied by Goldman [6] and Tan–Wong–Zhang [8,9] from the viewpoint of dynamics of the action of the mapping class group on the character variety. In their study Tan–Wong–Zhang introduced *Bowditch's Q-condition* (*BQ-condition*) for the characters, and showed that the characters satisfying the BQ-condition satisfy a variation of McShane's identity.

The following is the main theorem of this paper.

Theorem 1.1. *For $\rho \in \mathcal{R}_\theta^{\mathbb{R}}$, ρ satisfies the BQ-condition if and only if there is a polyhedron $Eh \subset \mathbb{H}^3$ and its side pairing with the following properties:*

1. *The polyhedron Eh is the geometric dual of a* 1*-dimensional simplicial complex induced from the associated sequence of elliptic generators.*
2. *By gluing the pairs of faces of Eh by the side pairing, we obtain a complete cone hyperbolic structure on M_θ whose holonomy representation is equal to ρ.*

Remark 1.2.

1. In [6], Goldman gave a characterization for the connected components of the real character variety for $\widetilde{\mathcal{R}}_\theta$, and then constructed a cone hyperbolic structures on T_θ for any character corresponding to an indefinite bilinear form. Then Tang–Wong–Zhang [8,9] studied the character variety by using generalized Markoff maps, where they showed a certain condition for a real character to satisfy the BQ-condition. The proof of Theorem 1.1 depends on their results.
2. Yamashita in collaboration with Tan made a series of experiments on the character variety, in which he examined when a character satisfies the BQ-condition (cf. [10]). On the other hand, we can generalize the condition for polyhedra used in Theorem 1.1 for non-real representations. A polyhedron is said to be *good* if it satisfies the condition. The author made a series of experiments when a representation has a good fundamental polyhedron. Based on the results of experiments, Yamashita and the author proposed a conjecture: A representation in $\mathcal{R}_\theta$ has a good fundamental polyhedron if and only if it satisfies the BQ-condition. Theorem 1.1 is a partial affirmative answer to the conjecture.

2. Preliminaries

2.1. Hyperbolic space

Throughout this paper, we use the upper half space model for the hyperbolic spaces $\mathbb{H}^2$ and $\mathbb{H}^3$, and identify the group of orientation-preserving isometries of $\mathbb{H}^2$ (resp. $\mathbb{H}^3$) with $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ (resp. $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm I\}$). Then the boundary at infinity $\partial\mathbb{H}^2$ (resp. $\partial\mathbb{H}^3$) is naturally identified