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# Hyperspaces of convex bodies of constant width ☆

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For every $n \geq 1$, let $cc(\mathbb{R}^n)$ denote the hyperspace of all non-empty compact convex subsets of the Euclidean space $\mathbb{R}^n$ endowed with the Hausdorff metric topology. For every non-empty convex subset $D$ of $[0, \infty)$ we denote by $cw_D(\mathbb{R}^n)$ the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex sets of constant width $d \in D$ and by $crw_D(\mathbb{R}^n)$ the subspace of the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ consisting of all pairs of compact convex sets of constant relative width $d \in D$. In this paper we prove that $cw_D(\mathbb{R}^n)$ and $crw_D(\mathbb{R}^n)$ are homeomorphic to $D \times \mathbb{R}^n \times Q$, whenever $D \neq \{0\}$ and $n \geq 2$, where $Q$ denotes the Hilbert cube. In particular, the hyperspace $cw(\mathbb{R}^n)$ of all compact convex bodies of constant width as well as the hyperspace $crw(\mathbb{R}^n)$ of all pairs of compact convex sets of constant relative positive width are homeomorphic to $\mathbb{R}^{n+1} \times Q$.



## 1. Introduction

Let $cc(\mathbb{R}^n)$, $n \geq 1$, denote the hyperspace of all non-empty compact convex subsets of $\mathbb{R}^n$ endowed with the topology induced by the Hausdorff metric:

$$\rho_H(Y, Z) = \max \left\{ \sup_{z \in Z} \|z - Y\|, \ \sup_{y \in Y} \|y - Z\| \right\}, \quad Y, Z \in cc(\mathbb{R}^n),$$

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where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$, i.e.,

$$\|x\|^2 = \sum_{i=1}^{n} x_i^2, \quad x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n.$$

In case $n = 1$, it is easy to see that $cc(\mathbb{R})$ is homeomorphic to $\mathbb{R} \times [0, 1)$ and for every $n \geq 2$, it is well known that $cc(\mathbb{R}^n)$ is homeomorphic to the *punctured Hilbert cube* $Q_0 := Q \backslash \{*\}$, where $Q := [0, 1]^\infty$ is the Hilbert cube (see [14, Theorem 7.3]).

By a *convex body* we mean a compact convex subset of $\mathbb{R}^n$ with non-empty interior. It is proved in [2, Corollary 3.11] that for every $n \geq 2$, the hyperspace $cb(\mathbb{R}^n)$ of all compact convex bodies, endowed with the Hausdorff metric topology, is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$. A compact convex set in $\mathbb{R}^n$ is said to be of constant width $d \geq 0$, if the distance between any two of its parallel support hyperplanes is equal to $d$ (see, e.g., [17, Chapter 7, §6]).

As a generalization of compact convex sets of constant width, H. Maehara introduced in [11] the concept of pairs of compact convex sets of constant relative width and showed that these pairs share certain properties of compact convex sets of constant width. A pair $(Y, Z) \in cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ is said to be of constant relative width $d \geq 0$, if

$$h_Y(u) + h_Z(-u) = d$$

for every $u \in \mathbb{S}^{n-1}$, where $h_Y$ and $h_Z$ denote the support functions of $Y$ and $Z$, respectively (see formula (2.1) below) and

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

is the unit sphere in $\mathbb{R}^n$.

For every non-empty convex subset $D$ of $[0, \infty)$ we denote by $cw_D(\mathbb{R}^n)$ the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex sets of constant width $d \in D$ and by $crw_D(\mathbb{R}^n)$ the subspace of the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ consisting of all pairs of compact convex sets of constant relative width $d \in D$. We shall use $cw(\mathbb{R}^n)$ and $crw(\mathbb{R}^n)$ for $cw_{(0,\infty)}(\mathbb{R}^n)$ and $crw_{(0,\infty)}(\mathbb{R}^n)$, respectively. The hyperspaces $cw(\mathbb{R}^n)$, $n \geq 2$, of all convex bodies of constant width in $\mathbb{R}^n$ were first considered in [4].

Note that if $D \subset (0, \infty)$, then every $A \in cw_D(\mathbb{R}^n)$ is a convex body. This does not hold in the hyperspace $crw_D(\mathbb{R}^n)$, i.e., there are pairs $(Y, Z)$ of compact convex sets of constant relative width $d > 0$, such that either $Y$ or $Z$ is not a convex body. For instance, the pair $(\mathbb{B}^n, \{0\})$ is of constant width 1, where

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

is the unit ball in $\mathbb{R}^n$, while $\{0\}$ is not a body.

Note also that if $D = \{0\}$, then the hyperspaces $cw_D(\mathbb{R}^n) = \{\{x\} \mid x \in \mathbb{R}^n\}$ and $crw_D(\mathbb{R}^n) = \{(\{x\}, \{x\}) \mid x \in \mathbb{R}^n\}$ are both homeomorphic to $\mathbb{R}^n$.

In case $n = 1$, it is easy to see that $cw_D(\mathbb{R})$ is homeomorphic to $D \times \mathbb{R}$ for every non-empty convex subset $D$ of $[0, \infty)$ and that $crw_D(\mathbb{R})$ is homeomorphic to $D \times \mathbb{R} \times [0, 1]$ for every non-empty convex subset $D \neq \{0\}$ of $[0, \infty)$ (see Propositions 3.1 and 4.2).

However, for every $n \geq 2$, the topological structure of $cw_D(\mathbb{R}^n)$ and $crw_D(\mathbb{R}^n)$ had remained unknown, except for the cases of convex sets $D$ of the form $[d_0, \infty)$ with $d_0 \geq 0$, for which it was proved in [5, Corollary 1.2] (relying on [5, Theorem 1.1]) that $cw_D(\mathbb{R}^n)$ is homeomorphic to the punctured Hilbert cube $Q_0$.

It is the purpose of this paper to give a complete description of the topological structure of the hyperspaces $cw_D(\mathbb{R}^n)$ and $crw_D(\mathbb{R}^n)$ for every $n \geq 2$ and every non-empty convex subset $D \neq \{0\}$ of $[0, \infty)$. Namely,