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Topological representation of lattice homomorphisms

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ABSTRACT

Wallman [13] proved that if $\mathbb{L}$ is a distributive lattice with **0** and **1**, then there is a T_1-space with a base (for closed subsets) being a homomorphic image of $\mathbb{L}$. We show that this theorem can be extended over homomorphisms. More precisely: if **NLat** denotes the category of normal and distributive lattices with **0** and **1** and homomorphisms, and **Comp** denotes the category of compact Hausdorff spaces and continuous mappings, then there exists a contravariant functor Ult : **NLat** $\to$ **Comp**. When restricted to the subcategory of Boolean lattices this functor coincides with a well-known Stone functor which realizes the Stone Duality. The functor $\mathcal{W}$ carries monomorphisms into surjections. However, it does not carry epimorphisms into injections. The last property makes a difference with the Stone functor. Some applications to topological constructions are given as well.

1. Basic facts and definitions

We shall consider *lattices* $(L, \leqslant)$ with zero and one, i.e. partially ordered sets with the smallest element **0** and the greatest element **1** in which for any two elements $x, y \in L$ there exist the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$. Since both infimum and supremum are unique, we get two binary operations $\vee$ and $\wedge$ which leads to an algebraic structure $\mathbb{L} = \langle L, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$. Immediately from definition of supremum and infimum we get

(a) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutativity),
(b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity),
(c) $x = x \vee (x \wedge y) = x \wedge (x \vee y)$ (absorption).

An algebraic structure $\mathbb{L}$ which satisfies conditions (a)–(c) is also called a lattice; see e.g. Theorem 1 in [2, p. 44]. We shall follow the common practice and denote the lattice and its underlying set by the

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same character. Properties (a) and (b) follow directly from the definition of supremum and infimum. The property (c) is in fact a consequence of the following equivalences:

$$x \leqslant y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x \vee y = y.$$

Together with commutativity and associativity the last equivalences imply

$$x \leqslant y \Longrightarrow x \wedge z \leqslant y \wedge z \text{ and } x \vee z \leqslant y \vee z$$

for all $x, y, z \in \mathbb{L}$. In particular we always have

$$x \wedge y \leqslant x \wedge (y \vee z) \text{ and } x \wedge z \leqslant x \wedge (y \vee z)$$

and finally

$$(x \wedge y) \vee (x \wedge z) \leqslant x \wedge (y \vee z).$$

The equality is not always true. A lattice $\mathbb{L}$ is called *distributive* whenever for any $x, y, z \in \mathbb{L}$ the following formula holds:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

All lattices considered here are assumed to be distributive and contain both $\mathbf{0}$ and $\mathbf{1}$.

We have the following trivial example of a distributive lattice: if S is a set then $\mathbb{L} = \langle \mathcal{P}(S), \cup, \cap, \emptyset, S \rangle$ is a distributive lattice. Of course, $\emptyset$ is the smallest and S is the greatest element of $\mathbb{L}$.

A lattice $\mathbb{K}$ is called a *sublattice* of a lattice $\mathbb{L}$ whenever $\mathbb{K}$ is a subset of $\mathbb{L}$, both the smallest and the greatest element of $\mathbb{L}$ belong to $\mathbb{K}$ and the lattice operations in $\mathbb{K}$ coincide with the lattice operations in $\mathbb{L}$.

It is obvious that a sublattice of a distributive lattice is distributive as well.

Important examples of distributive lattices are connected to topological spaces. If X is a topological space, then $\operatorname{cl} F$ denotes the closure of a set $F \subseteq X$. The set

$$\mathbb{C}\mathrm{l}(X) = \{F \subseteq X : F = \operatorname{cl} F\}$$

is a sublattice of the lattice $\mathcal{P}(X)$ of all subsets of X considered with the usual set operations. Clearly, $\mathbb{C}\mathrm{l}(X)$ is a distributive lattice. Let us introduce the following notion:

Definition 1.1. Let X be a topological space and $\mathbb{L} \subseteq \mathbb{C}\mathrm{l}(X)$ be a sublattice of $\mathbb{C}\mathrm{l}(X)$. Then $\mathbb{L}$ is called a *base lattice* if the set $\{X \setminus F \colon F \in \mathbb{L}\}$ is a base for the topology on X.

The base lattices will play important role in this paper.

We say that a lattice $\mathbb{L} = \langle L, \wedge, \vee, \mathbf{0}, \mathbf{1} \rangle$ is *normal* whenever it is distributive and for all $a, b \in \mathbb{L}$ with $a \wedge b = \mathbf{0}$ there exist $x, y \in \mathbb{L}$ such that

$x \vee y = \mathbf{1}$, and
$x \wedge a = y \wedge b = \mathbf{0}$.

Every Boolean algebra is a normal lattice. Let us recall that a distributive lattice $\mathbb{B}$ is a *Boolean algebra* (= Boolean lattice) if it is complementary, i.e. for every $a \in \mathbb{B}$ there exists an element $-a \in \mathbb{B}$ such that $a \vee -a = \mathbf{1}$ and $a \wedge -a = \mathbf{0}$. So, if $\mathbb{B}$ is a Boolean algebra then for $a \wedge b = \mathbf{0}$ it suffices to take $x = -a$ and $y = -b$.

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