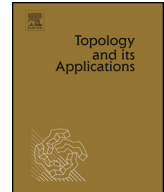




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Topological representation of lattice homomorphisms

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ABSTRACT

Wallman [13] proved that if \mathbb{L} is a distributive lattice with $\mathbf{0}$ and $\mathbf{1}$, then there is a T_1 -space with a base (for closed subsets) being a homomorphic image of \mathbb{L} . We show that this theorem can be extended over homomorphisms. More precisely: if \mathbf{NLat} denotes the category of normal and distributive lattices with $\mathbf{0}$ and $\mathbf{1}$ and homomorphisms, and \mathbf{Comp} denotes the category of compact Hausdorff spaces and continuous mappings, then there exists a contravariant functor $\text{Ult} : \mathbf{NLat} \rightarrow \mathbf{Comp}$. When restricted to the subcategory of Boolean lattices this functor coincides with a well-known Stone functor which realizes the Stone Duality. The functor \mathcal{W} carries monomorphisms into surjections. However, it does not carry epimorphisms into injections. The last property makes a difference with the Stone functor. Some applications to topological constructions are given as well.

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1. Basic facts and definitions

We shall consider *lattices* (L, \leq) with zero and one, i.e. partially ordered sets with the smallest element $\mathbf{0}$ and the greatest element $\mathbf{1}$ in which for any two elements $x, y \in L$ there exist the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$. Since both infimum and supremum are unique, we get two binary operations \vee and \wedge which leads to an algebraic structure $\mathbb{L} = \langle L, \wedge, \vee, \mathbf{0}, \mathbf{1} \rangle$. Immediately from definition of supremum and infimum we get

- (a) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutativity),
- (b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity),
- (c) $x = x \vee (x \wedge y) = x \wedge (x \vee y)$ (absorption).

An algebraic structure \mathbb{L} which satisfies conditions (a)–(c) is also called a lattice; see e.g. Theorem 1 in [2, p. 44]. We shall follow the common practice and denote the lattice and its underlying set by the

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same character. Properties (a) and (b) follow directly from the definition of supremum and infimum. The property (c) is in fact a consequence of the following equivalences:

$$x \leq y \iff x \wedge y = x \iff x \vee y = y.$$

Together with commutativity and associativity the last equivalences imply

$$x \leq y \implies x \wedge z \leq y \wedge z \text{ and } x \vee z \leq y \vee z$$

for all $x, y, z \in \mathbb{L}$. In particular we always have

$$x \wedge y \leq x \wedge (y \vee z) \text{ and } x \wedge z \leq x \wedge (y \vee z)$$

and finally

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z).$$

The equality is not always true. A lattice \mathbb{L} is called *distributive* whenever for any $x, y, z \in \mathbb{L}$ the following formula holds:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

All lattices considered here are assumed to be distributive and contain both $\mathbf{0}$ and $\mathbf{1}$.

We have the following trivial example of a distributive lattice: if S is a set then $\mathbb{L} = \langle \mathcal{P}(S), \cup, \cap, \emptyset, S \rangle$ is a distributive lattice. Of course, \emptyset is the smallest and S is the greatest element of \mathbb{L} .

A lattice \mathbb{K} is called a *sublattice* of a lattice \mathbb{L} whenever \mathbb{K} is a subset of \mathbb{L} , both the smallest and the greatest element of \mathbb{L} belong to \mathbb{K} and the lattice operations in \mathbb{K} coincide with the lattice operations in \mathbb{L} .

It is obvious that a sublattice of a distributive lattice is distributive as well.

Important examples of distributive lattices are connected to topological spaces. If X is a topological space, then $\text{cl } F$ denotes the closure of a set $F \subseteq X$. The set

$$\text{Cl}(X) = \{F \subseteq X : F = \text{cl } F\}$$

is a sublattice of the lattice $\mathcal{P}(X)$ of all subsets of X considered with the usual set operations. Clearly, $\text{Cl}(X)$ is a distributive lattice. Let us introduce the following notion:

Definition 1.1. Let X be a topological space and $\mathbb{L} \subseteq \text{Cl}(X)$ be a sublattice of $\text{Cl}(X)$. Then \mathbb{L} is called a *base lattice* if the set $\{X \setminus F : F \in \mathbb{L}\}$ is a base for the topology on X .

The base lattices will play important role in this paper.

We say that a lattice $\mathbb{L} = \langle L, \wedge, \vee, \mathbf{0}, \mathbf{1} \rangle$ is *normal* whenever it is distributive and for all $a, b \in \mathbb{L}$ with $a \wedge b = \mathbf{0}$ there exist $x, y \in \mathbb{L}$ such that

$$\begin{aligned} x \vee y &= \mathbf{1}, \text{ and} \\ x \wedge a &= y \wedge b = \mathbf{0}. \end{aligned}$$

Every Boolean algebra is a normal lattice. Let us recall that a distributive lattice \mathbb{B} is a *Boolean algebra* (= Boolean lattice) if it is complementary, i.e. for every $a \in \mathbb{B}$ there exists an element $-a \in \mathbb{B}$ such that $a \vee -a = \mathbf{1}$ and $a \wedge -a = \mathbf{0}$. So, if \mathbb{B} is a Boolean algebra then for $a \wedge b = \mathbf{0}$ it suffices to take $x = -a$ and $y = -b$.

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