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G-fibrations and twisted products

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We prove that the functor of twisted product $G \times_H -$ takes H-fibrations to G-fibrations when G is a compact metrizable (not necessarily Lie) group and H is its closed subgroup. This result is applied to the study of strong G-fibrations. In particular, we show that every G-map $E \to G/H$ is a strong G-fibration provided that E is a G-fibrant space.

0. Introduction

One of the remarkable facts of the theory of G-spaces is the following: if H is a closed subgroup of a *compact Lie* group G, then any G-map

$$p : E \to G/H$$

is a G-fibration. Here, by a G-fibration, we mean a G-map having the equivariant homotopy lifting property with respect to every G-space (see [10, p. 53]). This fact is well-known (as part of the theory of equivariant bundles) though it is not easy to find its explicit proof in the literature. One can recommend a detailed proof given in [6, Proposition 3.1] for the case of metrizable G-space E. As shown, in particular, in the present paper, this special case implies the general one. In fact, one of the aims of the paper is to prove that p is still a G-fibration even when G is any compact metrizable (not necessarily Lie) group.

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Naturally, the study of the properties of G-maps $E \to G/H$ is closely related to the study of the twisted products of the form $G \times_H F$ because of the elementary assertion (e.g., for a compact Hausdorff group G): a G-space E can be regarded as the twisted product $G \times_H F$ for some H-space F iff there exists a G-map $E \to G/H$.

The main result of the paper, which generalizes the fact mentioned at the beginning, can be formulated as follows: the functor of twisted product $G \times_H -$ takes H-fibrations to G-fibrations provided that G is a compact metrizable group and H is its closed subgroup (Theorem 6.4). Lemma 6.1 allows us to reduce the proof of this general result to the proof that the projection $G \to G/H$ is an H-fibration with respect to the action of H on G by conjugation. Therefore the key point of the paper is Proposition 6.3 which states that this canonical projection can be regarded as an H-fibration (in the above sense) for any compact metrizable group G (for compact Lie groups this is known, see the proof given on p. 266 in the paper of R. Lashof [15]).

At the end of the paper we consider some consequences of the main result. In particular, the fact that any G-map $p : E \to G/H$ is a G-fibration for a compact metrizable group G (Corollary 6.5) implies partial positive answers to Questions 4.4 and 4.5 stated by S. Antonyan in [4] (see Propositions 6.6 and 6.8(2)).

Other applications concern the strong G-fibrations introduced in [6]. Corollary 6.5 can be modified as follows: any G-map $p : E \to G/H$ is a strong G-fibration provided that E is a G-fibrant space (Proposition 6.7). We also show that the functor of twisted product $G \times_H -$ takes strong H-fibrations to strong G-fibrations if G/H is a metrizable G-ANE-space (Proposition 6.10).

1. Preliminaries

Throughout the paper the letter G will denote a compact Hausdorff group, while our main results concern the case of a compact metrizable group G; the unit element of G is denoted by e.

The foundations of the theory of G-spaces (also known as the theory of topological transformation groups) can be found in [8,10] and [17]. Below, for the convenience of the reader, we recall some well-known definitions and facts.

A *G-space* is a topological space X together with a fixed continuous (left) action $\cdot : G \times X \to X$, $(g, x) \mapsto g \cdot x$, of G on X. It is used to write simply gx instead of $g \cdot x$. Given G-spaces X and Y, a continuous map $f : X \to Y$ is called a *G-map* or an *equivariant map* if $f(gx) = gf(x)$ for all $(g, x) \in G \times X$. If a G-map f is a homeomorphism, we say that it is a *G-homeomorphism*. Clearly the G-spaces and the G-maps form a category which will be denoted by G-TOP.

Let X be a G-space. A subset $A \subset X$ is called *G-invariant* or a *G-subset* if $ga \in A$ for all $g \in G$ and $a \in A$. For $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *isotropy group* at x and the G-subset $G(x) = \{gx \mid g \in G\}$ is called the *G-orbit* of x. By a *free* G-space X we mean a space in which G acts *freely*, that is, $G_x = \{e\}$ for every $x \in X$.

Given a G-space X, the set of its G-orbits, endowed with the quotient topology, is called the *G-orbit space of X* and is denoted by X/G. The natural projection $\pi_X : X \to X/G$ (defined by $\pi_X(x) = G(x)$) is called the *G-orbit map* or the *G-orbit projection* of X; it is a G-map if we regard the G-orbit space X/G as a G-space with the trivial action of G.

If $f : X \to Y$ is a G-map, then there exists a unique continuous map $f/G : X/G \to Y/G$, called the map *induced by f*, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{f/G} & Y/G \end{array}$$

commutes. Clearly, f/G is defined by $(f/G)(G(x)) = G(f(x))$.

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