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Pseudo-Einstein manifolds ☆

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Among the studies on pseudo-Hermitian geometry of strictly pseudo-convex almost CR manifolds, we study especially the two kinds of pseudo-Einstein structures and related problems.

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1. Introduction

A *contact manifold* (M, η) is a smooth manifold M^{2n+1} together with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact manifold, two associated structures enrich the geometry. One is a Riemannian metric g compatible to η and we obtain a *contact Riemannian manifold* $(M; \eta, g)$. The other is a *pseudo-Hermitian and strictly pseudo-convex structure* (η, L) (or (η, J)), where L is the *Levi form* associated with an endomorphism J on D such that $J^2 = -I$. Here, J defines an almost CR structure $\mathcal{H} = \{X - iJX : X \in \Gamma(D)\}$, that is $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. We obtain a *strictly pseudo-convex, pseudo-Hermitian manifold (or almost CR manifold)* $(M; \eta, J)$. There is a one-to-one correspondence between the two associated structures by the relation

$$g = L + \eta \otimes \eta,$$

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where we denote by the same letter L the natural extension ($i_\xi L = 0$) of the Levi form to a $(0,2)$-tensor field on M. So, we treat contact Riemannian structures together with strictly pseudo-convex almost CR structures. For complex analytical considerations, it is desirable to have integrability of the almost complex structure J (on D). If this is the case, we speak of an *(integrable) CR structure* and of a *CR manifold*. Indeed, S. Webster [29,30] introduced the term *pseudo-Hermitian structure* for a CR manifold with a non-degenerate Levi-form. In the present paper, we treat the pseudo-Hermitian structure as an extension to the case of non-integrable $\mathcal{H}$.

There is a canonical affine connection in a non-degenerate CR-manifold, the so-called pseudo-Hermitian connection (or the Tanaka–Webster connection). S. Tanno [25] extends the Tanaka–Webster connection for strictly pseudo-convex almost CR manifolds (in which $\mathcal{H}$ is in general non-integrable). We call it the *generalized Tanaka–Webster connection*.

We may consider two kinds of the *pseudo-Hermitian Ricci curvature tensor* in a strictly pseudo-convex almost CR manifold $(M; \eta, J)$. One is defined by

$$\hat{\rho}_1(X,Y) = \text{trace of } \{V \mapsto \hat{R}(V,X)Y\},$$

which is called the *pseudo-Hermitian Ricci curvature tensor of the 1st kind*, the other is defined by

$$\hat{\rho}_2(X,Y) = \frac{1}{2} \text{ trace of } \{V \mapsto J\hat{R}(X,JY)V\},$$

where V is any vector field on M and X, Y are any vector fields orthogonal to ξ. We call this *pseudo-Hermitian Ricci curvature tensor of the 2nd kind*.

If the pseudo-Hermitian Ricci curvature tensor of the 1st kind (of the 2nd kind resp.) is a scalar (field) multiple of the Levi form in a strictly pseudo-convex almost CR manifold, then it is said to have the *pseudo-Einstein structure of the 1st kind (of the 2nd kind resp.)*. In Section 3, we study the relations of the two notions, their properties and examples. In particular, we prove that a pseudo-Einstein CR-manifold of the 1st kind admits necessarily the 2nd kind pseudo-Einstein structure (Corollary 8).

In Section 4, we study the generalized Chern–Morser curvature tensor C as a pseudo-conformal invariant in a strictly pseudo-convex almost CR manifold. We prove that for a strictly pseudo-convex almost CR-manifold M^{2n+1} ($n > 1$) with vanishing C, M is pseudo-Einstein of the 1st kind if and only if M is Sasakian space form (Theorem 21). Also, we prove that for a strictly pseudo-convex almost CR-manifold M^{2n+1} ($n > 1$) vanishing C, M is pseudo-Einstein of the 2nd kind if and only if M is of pointwise constant holomorphic sectional curvature for Tanaka–Webster connection (Theorem 22).

2. Preliminaries

We start by collecting some fundamental materials about contact Riemannian geometry and strictly pseudo-convex pseudo-hermitian geometry. All manifolds in the present paper are assumed to be connected, oriented and of class C^∞.

Contact Riemannian structures For a contact form η, there exists a unique vector field ξ, called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\eta(X) = g(X,\xi),\ d\eta(X,Y) = g(X,\varphi Y),\ \varphi^2 X = -X + \eta(X)\xi, \tag{1}$$

where X and Y are vector fields on M. From (1), it follows that

$$\varphi\xi = 0,\ \eta \circ \varphi = 0,\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2}$$

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