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Chart description for hyperelliptic Lefschetz fibrations and their stabilization

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ABSTRACT

Chart descriptions are a graphic method to describe monodromy representations of various topological objects. Here we introduce a chart description for hyperelliptic Lefschetz fibrations, and show that any hyperelliptic Lefschetz fibration can be stabilized by fiber-sum with certain basic Lefschetz fibrations.

1. Introduction

Chart descriptions were originally introduced in order to describe 2-dimensional braids in [8,9] (cf. [10]). In [13], a chart description for genus-one Lefschetz fibrations was introduced and an elementary proof of Matsumoto's classification theorem was given. At the third JAMEX meeting in Oaxaca, Mexico, 2004, the second author generalized it to a method describing any monodromy representation [11] and investigated genus-two Lefschetz fibrations as an application [12]. Here we introduce a chart description for hyperelliptic Lefschetz fibrations, and show that any hyperelliptic Lefschetz fibration can be stabilized by fiber-sum with certain basic Lefschetz fibrations.

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2. Lefschetz fibrations

Let M and B be compact, connected, and oriented smooth 4-manifold and 2-manifold, respectively. Let $f : M \to B$ be a smooth map with $\partial M = f^{-1}(\partial B)$. A critical point p is called a *Lefschetz singular point* of *positive type* (or of *negative type*, respectively) if there exist local complex coordinates z_1, z_2 around p and a local complex coordinate ξ around $f(p)$ such that f is locally written as $\xi = f(z_1, z_2) = z_1 z_2$ (or $\overline{z_1} z_2$, resp.). We call f a (smooth or differentiable) *Lefschetz fibration* if all critical points are Lefschetz singular points and if there exists exactly one critical point in the preimage of each critical value.

A *general fiber* is the preimage of a regular value of f. The *genus* of a Lefschetz fibration is defined to be the genus g of a general fiber. A *singular fiber* of *positive type* (or *negative type*, resp.) is the preimage of a critical value which contains a Lefschetz singular point of positive type (or negative type, resp.). A singular fiber is obtained by shrinking a simple loop, called a vanishing cycle, on a general fiber. In this paper we assume that a Lefschetz fibration is 'relatively minimal', i.e., all vanishing cycles are essential loops. We say that a singular fiber is of *type* I if the vanishing cycle is a non-separating loop. We say that a singular fiber is of *type* II_h for $h = 1, \ldots, [g/2]$ if the vanishing cycle is a separating loop which bounds a genus-h subsurface of the general fiber.

A singular fiber is of *type* I^+ if it is of type I and of positive type. Similarly *type* I^- and *type* II_h^+, *type* II_h^- for $h = 1, \ldots, [g/2]$ are defined. We denote by $n_0^+(f)$, $n_0^-(f)$, $n_h^+(f)$, and $n_h^-(f)$, the numbers of singular fibers of f of type I^+, I^-, II_h^+, and II_h^-, respectively. A Lefschetz fibration is called *irreducible* if every singular fiber is of type I, i.e., $n_h^+(f) = n_h^-(f) = 0$ for $h = 1, \ldots, [g/2]$. A Lefschetz fibration is called *chiral* or *symplectic* if every singular fiber is of positive type, i.e., $n_0^-(f) = n_h^-(f) = 0$ for $h = 1, \ldots, [g/2]$.

Let $f : M \to B$ be a Lefschetz fibration, and $\Delta = \{q_1, \ldots, q_n\}$ the set of critical values. Let $\rho : \pi_1(B \setminus \Delta, q_0) \to MC$ be the monodromy representation of f, where q_0 is a base point of $B \setminus \Delta$ and MC is the mapping class group of the fiber $f^{-1}(q_0)$. Consider a Hurwitz arc system for Δ, say $\mathcal{A} = (A_1, \ldots, A_n)$; each A_i is an embedded arc in B connecting q_0 and a point of Δ such that $A_i \cap A_j = \{q_0\}$ for $i \neq j$, and they appear in this order around q_0. When B is a 2-sphere or a 2-disk, the system $\mathcal{A}$ determines a system of generators of $\pi_1(B \setminus \Delta, q_0)$, say $(a_1, \ldots, a_n)$. We call $(\rho(a_1), \ldots, \rho(a_n))$ a *Hurwitz system* of f. For details on Hurwitz systems, refer to [1,7,15–17], etc.

Let ι be the mapping class of an involution of the fiber $f^{-1}(q_0)$ with $2g+2$ fixed points. The centralizer HMG of ι in MG is called the *hyperelliptic mapping class group* of $f^{-1}(q_0)$. A Lefschetz fibration is called *hyperelliptic* if the image of the monodromy representation ρ is included in HMG.

3. Main result

Let ζ_i $(i = 1, \ldots, 2g+1)$ be positive Dehn twists along the loops C_i $(i = 1, \ldots, 2g+1)$ illustrated in Fig. 1. The hyperelliptic mapping class group HMC of a genus-g Riemann surface is generated by $\zeta_1, \ldots, \zeta_{2g+1}$, and the following relations are defining relations (cf. [4]).

$$\zeta_i \zeta_j = \zeta_j \zeta_i \quad \text{if } |i-j| \geq 2, \tag{1}$$

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1} \quad \text{for } i = 1, \ldots, 2g, \tag{2}$$

$$\iota^2 = 1 \quad \text{where } \iota = \zeta_1 \cdots \zeta_{2g} \zeta_{2g+1}^2 \zeta_{2g} \cdots \zeta_1, \tag{3}$$

$$(\zeta_1 \cdots \zeta_{2g+1})^{2g+2} = 1, \tag{4}$$

$$\iota\, \zeta_i = \zeta_i\, \iota \quad \text{for } i = 1, \ldots, 2g+1. \tag{5}$$

Let σ_h be a positive Dehn twist along the loop S_h illustrated in Fig. 1. Then $\sigma_h = (\zeta_1 \cdots \zeta_{2h})^{4h+2}$ for $h = 1, \ldots, [g/2]$.