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Selection topologies

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ABSTRACT

Each weak selection for a space X defines an order-like relation, which in turn defines an open interval-like topology on X, called a selection topology. Spaces whose topology is determined by a collection of such selection topologies are called weak selection spaces. In this paper, we show that in the realm of second countable spaces, these are precisely the suborderable spaces. Motivated by this result, we refine an earlier result for weak orderability of second countable spaces.

1. Introduction

All spaces in this paper are assumed to be Hausdorff. For a space X, let $\mathscr{F}_2(X) = \{S \subset X : 1 \leq |S| \leq 2\}$, where $|S|$ is the cardinality of S. A map $\sigma : \mathscr{F}_2(X) \to X$ is a *weak selection* for X if $\sigma(S) \in S$ for every $S \in \mathscr{F}_2(X)$. Every weak selection σ generates an order-like relation $\preceq_\sigma$ on X [14, Definition 7.1] defined by $x \preceq_\sigma y$ if $\sigma(\{x, y\}) = x$. The relation $\preceq_\sigma$ is similar to a linear order being both total and antisymmetric, but is not necessarily transitive. Motivated by this relationship, we often write $x \prec_\sigma y$ to express that $x \preceq_\sigma y$ and $x \neq y$. A weak selection σ for X is *continuous* if it is continuous with respect to the Vietoris topology on $\mathscr{F}_2(X)$, which can be expressed by the property that for every $x, y \in X$ with $x \prec_\sigma y$, there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $s \prec_\sigma t$ for every $s \in U$ and $t \in V$, see [7, Theorem 3.1]. Continuity of a weak selection σ implies that all $\preceq_\sigma$-open intervals $(\leftarrow, x)_{\preceq_\sigma} = \{y \in X : y \prec_\sigma x\}$ and $(x, \rightarrow)_{\preceq_\sigma} = \{y \in X : x \prec_\sigma y\}$, $x \in X$, are open in X, but the converse is not necessarily true, see Section 2.

If σ is a continuous weak selection for X, then it remains continuous with respect to any other topology on X which is finer than the original one [7, Corollary 3.2]. Looking for a possible coarsest topology with this property, a natural topology $\mathscr{T}_\sigma$ on X was associated with σ, [7]. It was called a *selection topology*,

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and was defined following exactly the pattern of the usual open interval topology utilising the collection of $\preceq_\sigma$-open intervals $\mathscr{S}_\sigma = \{(\leftarrow, x)_{\preceq_\sigma}, (x, \rightarrow)_{\preceq_\sigma} : x \in X\}$ as a subbase. It was shown in [9] that $\mathscr{T}_\sigma$ is regular, and in [13] that $\mathscr{T}_\sigma$ is Tychonoff as well. Some pathological examples of continuous weak selections that are not continuous with respect to the selection topology they generate were given in [7] (see also [9,12]). Subsequently, answering a question of [10], it was shown in [13] that if there is a coarsest topology on a given set so that a weak selection defined on it is continuous, then this topology must be precisely the selection topology determined by the weak selection itself.

Regarding the distinction between the original topology and a selection topology, the following spaces $(X, \mathscr{T})$ were studied in [13]: *weakly determined by selections* if X admits a weak selection σ with $\mathscr{T} = \mathscr{T}_\sigma$; *determined by selections* if X admits a continuous weak selection σ with $\mathscr{T} = \mathscr{T}_\sigma$; and *strongly determined by selections* if X admits a continuous weak selection and $\mathscr{T} = \mathscr{T}_\sigma$ for every continuous weak selection σ for X. Every orderable space is determined by selections, and it was shown in [13, Example 3.4] that so also is the Sorgenfrey line, which is suborderable but not orderable. However, there are suborderable spaces which are not determined by selections, for instance such a space is $X = (0, 1) \cup \{2\} \subset \mathbb{R}$. As stated in an earlier version of [13],[1] it is unclear whether a (normal) space that is weakly determined by selections is also determined by selections. A weakly orderable space which is strongly determined by selections is orderable; every connected locally connected space which admits a continuous weak selection is strongly determined by selections (see [16]); and every compact space that admits a continuous weak selection is also strongly determined by selections. Answering a question of [10], it was shown in [13, Example 3.8] that there is a space which is strongly determined by selections and yet it is neither (locally) compact nor (locally) connected.

The idea of spaces determined by selections was generalised in [5]. For a set X and a family $\{\mathscr{T}_\alpha : \alpha \in \mathscr{A}\}$ of topologies on X, the *supremum topology* is the smallest topology on X, denoted by $\bigvee_{\alpha\in\mathscr{A}} \mathscr{T}_\alpha$, which contains all topologies $\mathscr{T}_\alpha$, $\alpha \in \mathscr{A}$. A topological space $(X, \mathscr{T})$ is a (*continuous*) *weak selection space* [5], if $\mathscr{T} = \bigvee_{\sigma\in\Sigma} \mathscr{T}_\sigma$ for some collection Σ of (continuous) weak selections for X. Some basic properties of these spaces, also several examples, were provided in [5].

We are now ready to state also the main purpose of this paper. One of our main results is Theorem 3.1 showing that (continuous) weak selection spaces are actually determined by only one (continuous) weak selection and a topological property which is almost identical to Purisch's characterisation of suborderability of metrizable space [17]. The only distinction there is about the large inductive dimension of totally disconnected subspaces of such spaces, which is unclear for (continuous) weak selection spaces, see Questions 1 and 2. In the special case of second countable spaces, the distinction is without a difference. Thus, Theorem 3.1 implies with ease that a second countable space is a (continuous) weak selection space if and only if it is suborderable, Corollary 3.8.

If a second countable space X is weakly determined by selections in sense of [13], then it is a weak selection space in sense of [5], and, by Corollary 3.8, suborderable as well. Recently, it was shown in [19] that every second countable suborderable space X (i.e., a subset of the real line) is determined by selections unless X has exactly two connected components one of which is compact while the other is an open interval. Thus, in the realm of second countable spaces, the difference between these spaces lies in these special subsets of the real line.

If $(X, \mathscr{T})$ is weakly determined by selections, then $\mathscr{T} = \mathscr{T}_\sigma$ for some (not necessarily continuous) weak selection σ for X. The selection σ is however not as arbitrary as it may look at first. It has the property that $\mathscr{T}_\sigma \subset \mathscr{T}$, hence being *separately continuous* in the terminology of [1,11]. In the last Section 4, motivated by this relationship, we refine [6, Theorem 1.1] by showing that a second countable space with a separately continuous weak selection is weakly orderable (Theorem 4.1).

[1] T. Nogura informed the author that [13, Example 3.6] provided by the referee of [13] is inaccurate.