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A multiple conjugation quandle and handlebody-knots

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We introduce a notion of multiple conjugation quandles and give some examples of it. A multiple conjugation quandle has a mixed structure of a group and a quandle. We see that the axioms of a multiple conjugation quandle are natural from the point of view of colorings for handlebody-knots, where a handlebody-knot is a handlebody embedded in the 3-sphere. A multiple conjugation quandle is not only used to define a coloring for handlebody-knots but also to provide a universal structure to define a quandle coloring invariant in a sense.

1. Introduction

A quandle [4,6] is an algebraic structure defined on a set with a binary operation. The algebraic structure is derived from the combinatorial structure of oriented knots, that is, Reidemeister moves for knot diagrams. Three axioms of a quandle correspond to the three Reidemeister moves, respectively. A coloring of a knot diagram is defined by associating an element of a quandle to each arc of a knot diagram, where two elements associated to an over-arc and one under-arc at a crossing determine the element associated to the other under-arc by the binary operation. Since each Reidemeister move induces a bijection between the sets of colorings of knot diagrams, any quandle gives an invariant of oriented knots, that is, the number of all colorings. The aim of this paper is to introduce an algebraic structure derived from the combinatorial structure of handlebody-knots, which is a generalization of a knot with respect to a genus of a handlebody.

A handlebody-knot is a handlebody embedded in the 3-sphere S^3. A genus 1 handlebody-knot is an embedding of a solid torus, which is equivalent to the knot represented by the core of the solid torus. A handlebody-knot can be represented by a spatial trivalent graph and its diagram such that the regular

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neighborhood of the spatial graph is the handlebody-knot. In other words, a handlebody-knot is a neighborhood equivalence class of a spatial graph (see [7] for neighborhood equivalence). In [1], the Reidemeister moves for handlebody-knots and a coloring invariant were introduced. This invariant is generalized in [2] and [3]. Roughly speaking, the invariants defined in [1,2], and [3] are invariants associated with $\mathbb{Z}/2\mathbb{Z}$, a cyclic group, and a (noncommutative) group, respectively. The invariant we define in this paper is a generalization of these invariants.

A multiple conjugation quandle, which is used to define the coloring invariant, has a mixed structure of a group and a quandle. The binary operation of a quandle structure is used to describe the coloring condition at a crossing, while the multiplication of a group structure is used to describe the coloring condition at a vertex. The associativity of the multiplication is realized as the IH-move, which is the local move on spatial trivalent graphs and causes the difference between handlebody-knots and spatial graphs. A multiple conjugation quandle is not only used to define a coloring for handlebody-knots but also to provide a universal structure to define a quandle coloring invariant. Since a handlebody-knot is unoriented, we give additive structure on a quandle with a good involution, which was introduced by Kamada [5] to define a coloring for unoriented knots and surface-knots. Then the resulting structure is the multiple conjugation quandle.

The homology theory for the multiple conjugation quandle is discussed in the ongoing joint work with Scott Carter and Masahico Saito. The theory unifies two homology theories for groups and quandles.

This paper is organized as follows. In Section 2, we introduce a notion of multiple conjugation quandles and give some examples of it. In Section 3, we define a coloring invariant by using a multiple conjugation quandle. In Section 4, we see that the axioms of a multiple conjugation quandle are derived from the Reidemeister moves for handlebody-knots.

2. A multiple conjugation quandle

We introduce a notion of multiple conjugation quandles and give some examples of it.

Definition 1. A *multiple conjugation quandle* X is the disjoint union of groups G_λ $(\lambda \in \Lambda)$ with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $a, b \in G_\lambda$, $a * b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x * e_\lambda = x$ and $x * (ab) = (x * a) * b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $(ab) * x = (a * x)(b * x)$, where $a * x, b * x \in G_\mu$ for some $\mu \in \Lambda$.

A *quandle* [4,6] is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $a \in X$, $a * a = a$.
- For any $a \in X$, the map $S_a : X \to X$ defined by $S_a(x) = x * a$ is a bijection.
- For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

We denote $S_b^n(a)$ by $a *^n b$. A *conjugation quandle* is a quandle defined on a group G by $a * b = b^{-1}ab$. Then a multiple conjugation quandle $X = \coprod_{\lambda \in \Lambda} G_\lambda$ is a quandle consisting of a family of conjugation quandles G_λ, where we note that the inverse of S_a is given by $S_{a^{-1}}$.

A multiple conjugation quandle X has a good involution $\rho : X \to X$ defined by $\rho(a) = a^{-1}$, where a good involution [5] is an involution of a quandle X satisfying

$$\rho(a * b) = \rho(a) * b, \qquad a * \rho(b) = a *^{-1} b$$