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Surface pole bracket polynomials of virtual knots and twisted knots

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ABSTRACT

Dye and Kauffman defined surface bracket polynomials for virtual links by use of surface states, and found a relationship between the surface states and the minimal genus of a surface in which a virtual link diagram is realized. They and Miyazawa independently defined a multivariable polynomial invariant of virtual links. This invariant is deeply related to the surface states. In this paper, we introduce the notion of surface pole bracket polynomials for link diagrams in closed surfaces, as a generalization of surface bracket polynomials by Dye and Kauffman. The polynomials induce the invariant of twisted links defined by the author before as a generalization of Dye, Kauffman and Miyazawa's polynomial invariant. Furthermore we discuss a relationship between curves in surface pole states and variables of the polynomial invariant.

1. Introduction

Virtual knot theory is a generalization of knot theory which is based on Gauss chord diagrams and link diagrams on closed oriented surfaces [7]. Virtual links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over closed oriented surfaces (cf. [2,6]). A twisted link defined by Bourgoin [1] is an extension of the notion of virtual links. Twisted links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over closed surfaces which are possibly non-orientable surfaces [1].

A *virtual link diagram* is a link diagram which may have *virtual crossings*, which are encircled crossings without over-under information. A *virtual link* is an equivalence class of a virtual link diagram by Reidemeister moves and virtual Reidemeister moves depicted in Figs. 1 and 2. We call these moves *generalized Reidemeister moves*.

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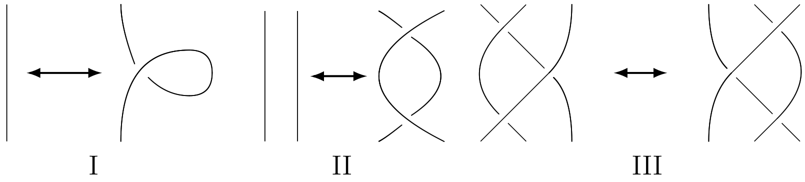

Fig. 1. Reidemeister moves.

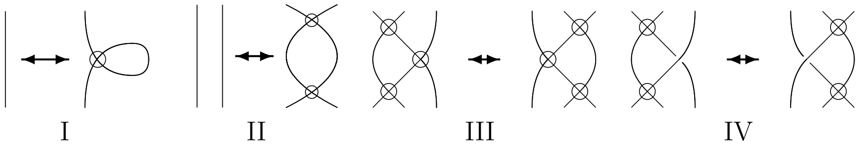

Fig. 2. Virtual Reidemeister moves.

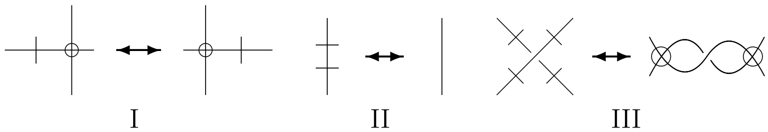

Fig. 3. Twisted Reidemeister moves.

A *twisted link diagram* is a virtual link diagram which may have *bars* on arcs. A *twisted link* is an equivalence class of a twisted link diagram by Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves in Figs. 1, 2 and 3. We call these moves *extended Reidemeister moves*.

Bourgoin introduced the Jones polynomials (f-polynomials) for twisted links and a group invariant called the twisted knot group [1]. The author introduced a twisted quandle for twisted links [5]. For a twisted link L, it is an interesting and important problem to determine an irreducible representative or to determine the minimum genus of a surface F in which a diagram of L is realized. Surface bracket polynomials of virtual links are defined by Dye and Kauffman [3] by use of surface states which are obtained from a link diagram in a closed oriented surface in which a diagram of L is realized.

The following conjecture is due to Kauffman and Przytycki.

Conjecture 1. *For a virtual knot L, if a diagram of L is realized in a surface of the minimal genus, then this fact is detected by the surface bracket polynomial.*

H.A. Dye and L.H. Kauffman [4], and Y. Miyazawa [8] independently, defined a multivariable polynomial invariant of virtual links, which we call the DKM polynomial. Dye and Kauffman showed that this invariant is deeply related to the surface states for link diagrams on closed oriented surfaces. In this paper, we introduce the notion of surface pole bracket polynomials for link diagrams in closed surfaces, as a generalization of surface bracket polynomials by Dye and Kauffman. The polynomials induce the invariant of twisted links defined by the author in [5] as a generalization of the DKM polynomial invariant. Then we discuss a relationship between curves in surface pole states and variables of the polynomial invariant.

2. Link diagram realizations of twisted links

An *abstract link diagram* is a pair (Σ, D_Σ) of a compact, possibly non-orientable surface Σ and a link diagram D_Σ in Σ such that $|D_\Sigma|$ is a deformation retract of Σ, where $|D_\Sigma|$ is the subset of Σ obtained from D_Σ by replacing each crossing with a 4-valent vertex. Two examples of abstract links are depicted in Fig. 4. The surface Σ in (ii) of the figure is a non-orientable surface.