Topology and its Applications 196 (2015) 510–521

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Three-dimensional braids and their descriptions

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ARTICLE INFO

Article history:
Received 24 December 2013
Received in revised form 5 March 2014
Accepted 5 March 2014
Available online 3 June 2015

MSC:
primary 57Q45
secondary 57M25

Keywords:
Three-dimensional braids
Branched coverings
Higher dimensional knots
Braid monodromy
Chart
Curtain

ABSTRACT

The notion of a braid is generalized into two and three dimensions. Two-dimensional braids are described by braid monodromies or graphics called charts. In this paper we introduce the notion of curtains, and show that three-dimensional braids are described by braid monodromies or curtains.

1. Introduction

Throughout this paper, we work in the PL category [8,15] and assume that all manifolds are oriented and m-manifolds embedded in $(m+2)$-manifolds are locally flat. We denote by D^2 the 2-disk and by B^m the m-disk. Let d be a positive integer and X_d a fixed set of d interior points of the 2-disk D^2.

For the product space $D^2 \times \Sigma^m$ of D^2 and an m-manifold Σ^m, we denote by $pr_1 : D^2 \times \Sigma^m \to D^2$ the first factor projection, and by $pr_2 : D^2 \times \Sigma^m \to \Sigma^m$ the second factor projection.

First we introduce the notion of a 3-dimensional braid.

Definition 1.

(1) A *3-dimensional braid* in $D^2 \times B^3$ (or over B^3) of degree d is a 3-manifold M embedded in $D^2 \times B^3$ such that (i) the restriction map $pr_2|_M : M \to B^3$ is a simple branched covering map of degree d branched along a link in B^3 and (ii) $\partial M = M \cap \partial(D^2 \times B^3) = X_d \times \partial B^3$.

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http://dx.doi.org/10.1016/j.topol.2015.05.012
0166-8641/

(2) A *3-dimensional braid* in $D^2 \times S^3$ (or over S^3) of degree d is a 3-manifold M embedded in $D^2 \times S^3$ such that (i) the restriction map $pr_2|_M : M \to S^3$ is a simple branched covering map of degree d branched along a link in S^3.

When we refer to a link, it may be the empty set. Refer to [1,2] for simple branched coverings.

More generally, we introduce the notion of a braided 3-manifold as follows. Let Σ^3 be a 3-manifold.

Definition 2. A *braided 3-manifold* in $D^2 \times \Sigma^3$ (or over Σ^3) of degree d is a 3-manifold M embedded in $D^2 \times \Sigma^3$ such that the restriction map $pr_2|_M : M \to \Sigma^3$ is a simple branched covering map of degree d and $\partial M = M \cap \partial(D^2 \times \Sigma^3) \subset \mathrm{int}D^2 \times \partial\Sigma^3$.

A 3-dimensional braid in $D^2 \times B^3$ is a braided 3-manifold in $D^2 \times B^3$ such that $\partial M = X_d \times \partial B^3$ and the branch set is a link in B^3. A 3-dimensional braid in $D^2 \times S^3$ is a braided 3-manifold in $D^2 \times S^3$ such that the branch set is a link in S^3.

Since any closed 3-manifold can be presented as a simple branched covering of S^3 branched along a link [7,13], our assumption that the branch set is a link is not so restrictive.

In this paper, we study how to describe 3-dimensional braids. We consider two methods, one is braid monodromies and the other is curtain descriptions. The idea of the curtain description was introduced in [3], and some examples were shown in [3,4]. However, existence of a curtain for any 3-dimensional braid was not shown. The main purpose of this paper is to show how to construct a curtain.

The first author was supported by the Ministry of Education, Science and Technology (MEST) and the Korean Federation of Science and Technology Societies (KOFST) during the initial phases of this work. The second author was supported by JSPS KAKENHI Grant Number 21340015.

2. 2-Dimensional braids, braid monodromies and charts

Before going to the case of 3-dimension in the next section, we quickly recall the notions of 2-dimensional braids, braid monodromies and charts. For the precise definitions and details, refer to [5,12]. The reader who is familiar with these notions may skip this section.

Let Σ^2 be a surface.

Definition 3. A *braided surface* in $D^2 \times \Sigma^2$ (or over Σ^2) of degree d is a surface S embedded in $D^2 \times \Sigma^2$ such that the restriction map $pr_2|_S : S \to \Sigma^2$ is a simple branched covering map of degree d and $\partial S = S \cap \partial(D^2 \times \Sigma^2) \subset \mathrm{int}\, D^2 \times \partial\Sigma^2$.

(1) A *2-dimensional braid* in $D^2 \times B^2$ (or over B^2) is a braided surface in $D^2 \times B^2$ such that $\partial S = X_d \times \partial B^2$.
(2) A *2-dimensional braid* in $D^2 \times S^2$ (or over S^2) is a braided surface in $D^2 \times S^2$.

Definition 4. Two 2-dimensional braids S and S' in $D^2 \times B^2$ are said to be *equivalent* if there is an ambient isotopy $\{h_s : D^2 \times B^2 \to D^2 \times B^2\}_{s\in[0,1]}$ such that

(1) $h_0 = \mathrm{id}$ and $h_1(S) = S'$,
(2) there is an ambient isotopy $\{\underline{h}_s : B^2 \to B^2\}_{s\in[0,1]}$ with $\underline{h}_s \circ pr_2 = pr_2 \circ h_s$ for each $s \in [0,1]$, and
(3) for each $s \in [0,1]$, the restriction map of h_s to $D^2 \times \partial B^2$ is the identity map.

Moreover, if $\underline{h}_s = \mathrm{id} : B^2 \to B^2$ for each $s \in [0,1]$, then we say that S and S' are *isomorphic*.