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An estimate of the Rasmussen invariant for links and the determination for certain links

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A R T I C L E I N F O

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Improving the slice-Bennequin inequality shown by Rudolph, we estimate some knot or link invariants, especially the knot invariant defined by Ozsváth and Szabó and the Rasmussen invariant for links introduced by Beliakova and Wehrli. Our argument implies a combinatorial proof of the slice-Bennequin inequality for links. Furthermore we determine such invariants for negative links and certain pretzel knots.

1. Introduction

A *link* is a closed oriented 1-manifold smoothly embedded in the 3-sphere S^3 and a *knot* is a link with one component.

Let L be a link. The *slice Euler characteristic* of the link L is the maximum Euler characteristic for an oriented compact 2-manifold without closed component, which is smoothly embedded in the 4-ball B^4 with boundary $L \subset S^3 = \partial B^4$. We denote this invariant by $\chi_s(L)$. We note the 2-manifold is not assumed to be connected. Using the gauge theory, Rudolph estimated the slice Euler characteristics for links as follows. Let D be an oriented link diagram. We respectively denote by $x(D)$, $x_+(D)$, $x_-(D)$, and $O(D)$ the number of crossings, the number of positive crossings, the number of negative crossings, and the number of Seifert circles of D. The *writhe* $w(D)$ is the value defined by $x_+(D) - x_-(D)$. In [17] Rudolph showed that the following inequality holds for any link L and any diagram D_L of L:

$$\chi_s(L) \leq O(D_L) - w(D_L). \tag{1}$$

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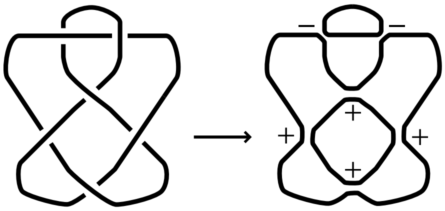

Fig. 1. An example of a link diagram and its Seifert circles.

This inequality is called the *slice-Bennequin inequality*. Furthermore, he and Nakamura independently showed in [18,12] that the following equality holds for any positive link L and any positive diagram D_L of L:

$$\chi_s(L) = O(D_L) - x(D_L). \tag{2}$$

A *positive (link) diagram* is a link diagram without negative crossing, and a *positive link* is a link which has a positive diagram. We note the trivial link is positive.

A Seifert circle is a *strongly negative circle* if it is adjacent to at least two negative crossings but adjacent to no positive crossings. A Seifert circle is a *non-negative circle* if it is not a strongly negative circle. For example, the trivial diagram of a trivial knot has a non-negative circle. The blackboard diagram of a (-1)-framed trivial knot, that is, the knot diagram with a single negative crossing but no positive crossings, has two non-negative circles. A knot diagram illustrated on the left of Fig. 1 has three Seifert circles as illustrated on the right. The top circle is strongly negative and the concentric circles at the bottom are non-negative. For a given oriented link diagram D, we denote by $O_<(D)$ and $O_\geq(D)$ the number of strongly negative circles and non-negative circles respectively. It is clear that the equality $O(D) = O_<(D) + O_\geq(D)$ holds.

In [5] the author restated the inequality

$$\chi_s(L) \leq O_\geq(D_L) - O_<(D_L) - w(D_L), \tag{3}$$

which Rudolph announced in [17], and completely proved that it holds for any link L and any diagram D_L of L. By an argument similar to that for the proof of this inequality, we improve the above inequalities (1) and (3) as follows.

For a link L and a diagram D_L of L, we eliminate all negative crossings in the same manner as the Seifert algorithm, and denote the obtained diagram by D_L^{0+}. This diagram is positive. All strongly negative circles of D_L appear as split trivial components in D_L^{0+}. We remove such circles from D_L^{0+}, and denote the obtained diagram by D_L^+. The new diagram D_L^+ is empty or positive. The diagrams D_L^{0+} and D_L^+ are defined uniquely from D_L. In this article, we denote by $l_0(D_L)$ and $l(D_L)$ the number of the split components of the link represented by the diagrams D_L^{0+} and D_L^+ respectively. It is clear that $l_0(D_L) = l(D_L) + O_<(D_L)$.

For example, let D be the blackboard diagram of a (-1)-framed trivial knot. The positive diagram D^{0+} of D represents a trivial 2-component link, and the positive diagram D^+ is also. Then we have $l_0 = l = 2$. The positive diagram D_K^+ of the knot diagram D_K illustrated on the left of Fig. 1 represents the $(2,4)$-torus link, then we have $l_0 = 2$ and $l = 1$. For the knot diagram D_L illustrated on the left of Fig. 2, the positive diagrams D_L^{0+} and D_L^+ are illustrated on the middle and right of Fig. 2. In this case, we have $l_0 = 3$ and $l = 2$.

Theorem 1.1. *Let L be a link and D_L be a diagram of L. If the link L is not splittable, then the following inequality holds:*

$$\begin{aligned}\chi_s(L) &\leq O_\geq(D_L) - O_<(D_L) - w(D_L) - 2(l(D_L) - 1)\\ &= O(D_L) - w(D_L) - 2(l_0(D_L) - 1).\end{aligned}$$