Topology and its Applications 196 (2015) 575–593

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On 4-dimensional universe for every 3-dimensional manifold

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A R T I C L E   I N F O

Article history:
Received 8 October 2013
Received in revised form 20 February 2014
Accepted 20 February 2014
Available online 27 May 2015

MSC:
57M27
57N13
57N35

Keywords:
Universe
Punctured universe
Topological index
3-manifold
Punctured 3-manifold
Type 1 embedding
Type 2 embedding
Signature theorem

A B S T R A C T

A boundary-less connected oriented 4-manifold is called a universe for every 3-manifold if every closed connected oriented 3-manifold is embedded in it, and a punctured universe if every punctured 3-manifold is embedded in it, which is known to be an open 4-manifold. We introduce types 1, 2 and full universes as refined notions of a universe and a punctured universe and investigate some relationships among them. After introducing some topological invariants for every (possibly non-compact) oriented 4-manifold which we call the topological indexes, we show infinity and independence on some topological indexes of every universe.

1. Introduction

Throughout this paper, by a *closed 3-manifold* we mean a closed connected oriented 3-manifold and by a *punctured 3-manifold* a punctured manifold of a closed connected oriented 3-manifold. Then we know that for every compact oriented 4-manifold, there is a closed 3-manifold whose punctured 3-manifold is not embeddable in it (see [4][2]) and hence any oriented 4-manifold with every punctured 3-manifold embedded must be non-compact. This motivates us to put the following definition:

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[1] This work was supported by JSPS KAKANHI Grant Number 24244005.

[2] The non-orientable version is also known in [11], but we do not discuss it here. Also, by an embedding we will mean a smooth or piecewise-linear embedding.

http://dx.doi.org/10.1016/j.topol.2015.05.035

Definition. A *universe* is an open connected oriented 4-manifold U with every closed 3-manifold M embedded. A *punctured universe* is an open connected oriented 4-manifold U with every punctured 3-manifold M^0 embedded.

Then we ask a question: *What topological shapes a universe and a punctured universe have?*

In this question, we introduce the following topological indexes

$$\hat{\beta}_d(Y)(d=1,2),\ \delta(Y),\ \delta_i(Y)\,(i=0,1,2),\ \rho(Y),\ \rho_i(Y)\,(i=0,1,2)$$

of every (possibly, non-compact) oriented 4-manifold Y, which are obtained from homological arguments and are topological invariants of Y with values taken in $\{0,1,2,\ldots,+\infty\}$. We apply these invariants to a punctured universe, a universe and their refined universes, namely types 1, 2 and full universes to obtain our main result (Theorem 3.3) which is stated as follows:

For a punctures universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ is $+\infty$. Further, in every case, there is a punctured spin universe U with the other topological indexes taken 0. For a type 1 universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ is $+\infty$. We have always $\hat{\beta}_1(U) \geqq 1$, but in the case of $\rho_1(U) = +\infty$, we can add the condition that $\hat{\beta}_1(U) = +\infty$. Further, in every case, there is a type 1 spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ taken 0.
For a type 2 universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta_2(U)$ is $+\infty$. Further, in every case, there is a type 2 spin universe U with the other topological index taken 0.
For a universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$, $\rho(U)$ is $+\infty$. In the case of $\rho(U) = +\infty$, we can add the condition that $\hat{\beta}_1(U) = +\infty$. Further, in every case, there is a spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta(U)$ and $\rho(U)$ taken 0.
For a full universe U, we show that one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$ is $+\infty$. We have always $\hat{\beta}_1(U) \geqq 1$. Further, in every case, there is a full spin universe U with the other topological index on $\hat{\beta}_2(U)$ and $\delta(U)$ taken 0.

In Section 2, we introduce types 1, 2 and full universes as refined notions of a universe and a punctured universe. We explain some relationships among them in Theorem 2.1. In Section 3, the topological indexes of every oriented 4-manifold are defined and our main result (Theorem 3.3) is stated. The existence part of universes in our main result (Theorem 3.3) is shown in this section with some examples. In Section 4, we establish a non-compact 4-manifold version of the signature theorem for an infinite cyclic covering of a compact oriented manifold given in [3], which is needed to prove the infinity of some topological indexes stated in Theorem 3.3. In Section 5, we introduce a notion of a loose embedding needed as a tool connecting an embedding argument with an argument of an infinite cyclic covering. In Section 6, we complete the proof of Theorem 3.3.

2. Types 1, 2 and full universes as refined notions of a universe and a punctured universe

Let $\mathbb{M}$ be the set of closed 3-manifolds M, and $\mathbb{M}^0$ the set of punctured 3-manifolds M^0. It is useful to denote the members of $\mathbb{M}$ and $\mathbb{M}^0$ by M_i $(i = 1, 2, 3, \ldots)$ and M_i^0 $(i = 1, 2, 3, \ldots)$, respectively. For a connected open oriented 4-manifold U, we note that there are two types of embeddings $k : M \to U$. An embedding $k : M \to U$ is *of type 1* if $U \backslash k(M)$ is connected, and *of type 2* if $U \backslash k(M)$ is disconnected (see Fig. 1). If there is a type 1 embedding $k : M \to U$, then there is an element $x \in H_1(U; Z)$ with the