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Naturally reductive homogeneous real hypersurfaces in a nonflat complex space form

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ABSTRACT

In this paper, in the class of homogeneous real hypersurfaces in a nonflat complex space form we study naturally reductive Riemannian homogeneous spaces having nonnegative sectional curvature.

1. Introduction

Theory of Riemannian homogeneous spaces is one of the most interesting objects in differential geometry. In particular, we pay particular attention to naturally reductive Riemannian homogeneous spaces M. Let $M = G/K$ be a Riemannian homogeneous space with Riemannian metric g, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. We call $M = G/K$ *reductive* if there is an Ad_K-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ satisfying

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}. \tag{1.1}$$

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This is called a *reductive decomposition*. A Riemannian homogeneous space M is said to be *naturally reductive* if it is naturally reductive with respect to some transitive Lie subgroup of the isometry group of M. Here, $M = G/K$ is *naturally reductive* with respect to G if there is a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that

$$g([X, Z]_{\mathfrak{m}}, Y) + g(Z, [X, Y]_{\mathfrak{m}}) = 0 \tag{1.2}$$

holds for all $X, Y, Z \in \mathfrak{m}$. Note that $[\ ,\ ]_{\mathfrak{m}}$ denotes the canonical projection onto $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This notion gives us some geometric properties. For example, it is known that every geodesic $\gamma = \gamma(s)$ on each naturally reductive Riemannian homogeneous space M is a homogeneous curve, namely the curve γ is an orbit of some one-parameter subgroup of the isometry group $\mathrm{I}(M)$ of M. In fact, a geodesic $\gamma = \gamma(s)$ with $\gamma(0) = o$ is an orbit of the one-parameter subgroup generated by $X := \dot{\gamma}(0) \in \mathfrak{m}$, where we canonically identify $\mathfrak{m}$ with the tangent space T_oM at the origin o. We refer to [6] for details. A Riemannian manifold all of whose geodesics are homogeneous is called a *geodesic orbit space* or a *GO-space*. Naturally reductive homogeneous spaces are GO-spaces, but the converse does not hold. We refer to, for examples, [4,13].

It is known that every naturally reductive Riemannian homogeneous space with nonpositive sectional curvature is a Riemannian symmetric space (see [5]). On the contrary, there exist naturally reductive Riemannian homogeneous spaces with nonnegative sectional curvature which are *not* Riemannian symmetric spaces.

In this paper, we study naturally reductive Riemannian homogeneous spaces with nonnegative sectional curvature in the class of homogeneous real hypersurfaces M^{2n-1} of a nonflat complex space form $\widetilde{M}_n(c)$, $n \geqq 2$. Here, the ambient space $\widetilde{M}_n(c)$ is holomorphically isometric to either a complex projective space $\mathbb{C}P^n(c)$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ according as c is positive or negative.

A real hypersurface of $\widetilde{M}_n(c)$ is said to be *homogeneous* if it is an orbit of a subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space $\widetilde{M}_n(c)$. Homogeneous real hypersurfaces in $\widetilde{M}_n(c)$ have been classified (see Theorems A and B). Typical examples of homogeneous real hypersurfaces are hypersurfaces of type (A), consisting of geodesic spheres, tubes around totally geodesic complex submanifolds, and horospheres. The purpose of this paper is to prove the following:

Theorem. *Let M^{2n-1} be a connected real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ with $n \geqq 2$ through an isometric immersion. Then we have the following statements* (1), (2) *and* (3).

(1) *The following four conditions are mutually equivalent:*
  1_a) *M is locally congruent to a naturally reductive Riemannian homogeneous space, and is locally congruent to a homogeneous real hypersurface of $\widetilde{M}_n(c)$;*
  1_b) *M is locally congruent to a hypersurface of type* (A) *of $\widetilde{M}_n(c)$;*
  1_c) *Every geodesic $\gamma = \gamma(s)$ on M, considered as a curve in $\widetilde{M}_n(c)$, has constant first curvature $\kappa(s) := \|\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\|$ along the curve γ, where $\widetilde{\nabla}$ is the Riemannian connection of $\widetilde{M}_n(c)$;*
  1_d) *M is locally congruent to a GO-space, and is locally congruent to a homogeneous real hypersurface of $\widetilde{M}_n(c)$.*

(2) *When $c > 0$, every real hypersurface M satisfying one of the above four conditions 1_a), 1_b), 1_c), and 1_d) has nonnegative sectional curvature.*

(3) *When $c < 0$, a real hypersurface M satisfies one of the above four conditions 1_a), 1_b), 1_c), and 1_d), and has nonnegative sectional curvature if and only if M is locally congruent to a geodesic sphere $G(r)$ of radius r with $0 < r \leqq \log 3/\sqrt{|c|}$ in $\mathbb{C}H^n(c)$.*