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HOMFLY polynomials for 3-component links with v-span 4

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A R T I C L E I N F O

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A B S T R A C T

The HOMFLY polynomial $P_L(v, z)$ of a link L is a well known link invariant with two variables v and z, which generalizes the Jones polynomial. The v-span of the HOMFLY polynomial is the difference between the maximum and the minimum degrees of the polynomial in the variable v. We show the HOMFLY polynomial of the 3-component link with v-span 4 is determined by the Jones polynomial.

1. Introduction

In this paper, we study the HOMFLY polynomials [5,9,12] of 3-component links.

The HOMFLY polynomial $P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ of an oriented link L is an invariant of the isotopy type of L, which is defined by the following formulas:

1. $P(U; v, z) = 1$;
2. $v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z)$,

where U is the trivial knot and L_+, L_- and L_0 are three links that are identical except near one point where they are as in Fig. 1. We call (L_+, L_-, L_0) a skein triple. The second formula is called the skein relation for the HOMFLY polynomial.

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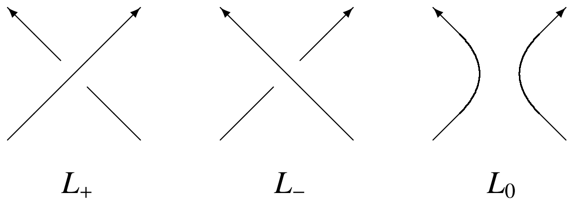

Fig. 1. A skein triple.

The reduced polynomial $P(L;1,z)$ of L is called the Conway polynomial [2] of L and is denoted by $\nabla(L;z)$. The Alexander polynomial $\Delta(L;t)$ [1] of L is equivalent to the Conway polynomial of L and is expressed as $\nabla(L;t^{1/2}-t^{-1/2}) = P(L;1,t^{1/2}-t^{-1/2})$. The Jones polynomial $V(L;t)$ [6] of L is defined as the reduced polynomial $P(L;t,t^{1/2}-t^{-1/2})$.

The HOMFLY polynomial $P(L;v,z)$ of L can be written as

$$P(L;v,z) = \sum_{j\in\mathbb{Z}} C_j(L;z)v^j,$$

where $C_j(L;z) \in \mathbb{Z}[z^{\pm 1}]$. Each Laurent polynomial $C_j(L;z)$ is called a *coefficient polynomial* of $P(L;v,z)$ in v. Then, two integers $\max\deg_v P(L;v,z)$ and $\min\deg_v P(L;v,z)$ are defined as follows:

$$\max\deg_v P(L;v,z) = \max\{j; C_j(L;z) \neq 0\},$$
$$\min\deg_v P(L;v,z) = \min\{j; C_j(L;z) \neq 0\}.$$

By using these values, the non-negative integer denoted by $v\text{-span}P(L;v,z)$ is defined in the following.

$$v\text{-span}P(L;v,z) = \max\deg_v P(L;v,z) - \min\deg_v P(L;v,z).$$

It is called the *v-span* of the HOMFLY polynomial $P(L;v,z)$.

Considering a relationship between the HOMFLY and the Jones polynomials for the link, we see that the HOMFLY polynomial dominates the Jones polynomial since the Jones polynomial is defined to be a reduced polynomial from the HOMFLY polynomial. The converse of the fact is not necessarily true. We know that there are a lot of pairs of links such that their HOMFLY polynomials are distinct even though their Jones polynomials coincide. The following result shows that the converse is true for some links.

Theorem 1.1. *Let L be an oriented 3-component link. If $v\text{-span}P(L;v,z) = 4$, then $P(L;v,z)$ is determined by $V(L;t)$.*

The following corollary, which is immediately obtained from Theorem 1.1 and Lemma 2.8, gives a concrete example of the link whose Jones polynomial controls its HOMFLY polynomial.

Corollary 1.2. *([3]) The HOMFLY polynomial of the 3-component link with braid index 3 is determined by its Jones Polynomial.*

Remark 1.3. The 3-component link with braid index 3 is equivalent to the closed pure 3-braid.

The proof of Theorem 1.1 is given in Section 5. Other sections are devoted to preparation for the proof. In Section 2, we discuss degrees of polynomials derived from the HOMFLY polynomial for the 3-component link with v-span 4. In Section 3, we compute the HOMFLY and the Conway polynomials for torus links of a certain type. In Section 4, we study the structure of the HOMFLY polynomial for the link.

Unless otherwise specified in the following, we suppose that knots and links are oriented.