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A homotopy classification of two-component spatial graphs up to neighborhood equivalence

Atsuhiko Mizusawa [a,*], Ryo Nikkuni [b,1]

[a] *Department of Mathematics, Faculty of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan*
[b] *Department of Mathematics, School of Arts and Sciences, Tokyo Woman's Christian University, 2-6-1 Zempukuji, Suginami-ku, Tokyo 167-8585, Japan*

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A B S T R A C T

A neighborhood homotopy is an equivalence relation on spatial graphs which is generated by crossing changes on the same component and neighborhood equivalence. We give a complete classification of all 2-component spatial graphs up to neighborhood homotopy by the elementary divisor of a linking matrix with respect to the first homology group of each of the connected components. This also leads a kind of homotopy classification of 2-component handlebody-links.

1. Introduction

Throughout this paper we work in the piecewise linear category. An embedding of a graph into the 3-sphere $\mathbb{S}^3$ is called a *spatial embedding* of the graph and the image is called a *spatial graph*. We say that two spatial graphs G and G' are *ambient isotopic* if there exists an orientation-preserving self-homeomorphism Φ on $\mathbb{S}^3$ such that $\Phi(G) = G'$. A graph is said to be *planar* if there exists an embedding of the graph into

* Corresponding author.
E-mail addresses: a_mizusawa@aoni.waseda.jp (A. Mizusawa), nick@lab.twcu.ac.jp (R. Nikkuni).
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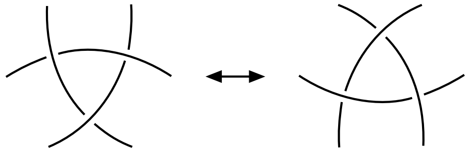

Fig. 1.1. Delta move.

the 2-sphere, and a spatial embedding of a planar graph is said to be *trivial* if it is ambient isotopic to an embedding of the graph into a 2-sphere in $\mathbb{S}^3$. Such an embedding is unique up to ambient isotopy [3]. On the other hand, let us denote the regular neighborhood of a spatial graph G in $\mathbb{S}^3$ by $N(G)$. Then, two spatial graphs G and G' are said to be *neighborhood equivalent* if there exists an orientation-preserving self-homeomorphism Φ on $\mathbb{S}^3$ such that $\Phi(N(G)) = N(G')$ [11]. Note that ambient isotopic two spatial graphs are homeomorphic as abstract graphs, but neighborhood equivalent two spatial graphs are not always homeomorphic. For a spatial graph G and an edge e of G which is not a loop, we call the spatial graph obtained from $G - \operatorname{int} e$ by identifying the end vertices of e the *edge contraction* of G along e. A *vertex splitting* is the reverse of an edge contraction. Then it is known that two spatial graphs are neighborhood equivalent if they are transformed into each other by edge contractions, vertex splittings and ambient isotopies [2].

Two oriented links are said to be *link homotopic* if they are transformed into each other by crossing changes on the same component and ambient isotopies. It is well known that two oriented 2-component links are link homotopic if and only if they have the same linking number [5]. Our purpose in this article is to generalize this fact to 2-component spatial graphs from a viewpoint of neighborhood equivalence. We introduce the notion of *neighborhood homotopy* on spatial graphs as an equivalence relation which is generated by crossing changes on the same component and neighborhood equivalence; that is, two spatial graphs G and G' are neighborhood homotopic if they are transformed into each other by crossing changes between edges which belong to the same component, edge contractions, vertex splittings and ambient isotopies. Note that in the case of oriented links, neighborhood homotopy coincides with link homotopy. Moreover, we also introduce another equivalence relation on spatial graphs as follows. A *Delta move* is a local move on a spatial graph as illustrated in Fig. 1.1 [4,8]. We say that two spatial graphs are *Delta neighborhood equivalent* if they are transformed into each other by Delta moves, edge contractions, vertex splittings and ambient isotopies.

In [6], the first author introduced a sequence of invariant nonnegative integers for 2-component spatial graphs under neighborhood equivalence as follows. Let $G = G_1 \cup G_2$ be a 2-component spatial graph. Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_m\}$ be a basis of $H_1(G_1;\mathbb{Z})$ and $\mathcal{W} = \{w_1, w_2, \ldots, w_n\}$ a basis of $H_1(G_2;\mathbb{Z})$. Let $M_G(\mathcal{Z}, \mathcal{W})$ be the (m, n)-matrix whose (i, j)-entry is the *linking number* $\operatorname{lk}(z_i, w_j)$ in $\mathbb{S}^3$. Then the sequence of elementary divisors $d_1, d_2, \ldots, d_l$ ($d_i \in \mathbb{Z}_{>0}$, $d_i | d_{i+1}$ $(i = 1, 2, \ldots, l-1)$) of $M_G(\mathcal{Z}, \mathcal{W})$ is an invariant under neighborhood equivalence. We define $\operatorname{Lk}(G_1, G_2)$ by the sequence $\{d_1, d_2, \ldots, d_l\}$ if $l \geq 1$ and otherwise 0. Now we state our main theorem.

Theorem 1.1. *Let $G = G_1 \cup G_2$ and $G' = G'_1 \cup G'_2$ be two 2-component spatial graphs satisfying with $H_1(G_i;\mathbb{Z}) \cong H_1(G'_i;\mathbb{Z})$ $(i = 1, 2)$. Then the following are equivalent.*

1. *G and G' are neighborhood homotopic.*
2. *G and G' are Delta neighborhood equivalent.*
3. $\operatorname{Lk}(G_1, G_2) = \operatorname{Lk}(G'_1, G'_2)$.

A *handlebody-link* in $\mathbb{S}^3$ is the image of an embedding of mutually disjoint handlebodies into $\mathbb{S}^3$. Two handlebody-links L and L' are said to be *equivalent* if there exists an orientation-preserving self-homeomorphism Φ on $\mathbb{S}^3$ such that $\Phi(L) = L'$. Note that two handlebody-links are equivalent if and only