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On knots with no 3-state

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A B S T R A C T

A state of a virtual knot diagram D is a disjoint union of circles obtained from D by splicing all real crossings. For each positive integer n, we denote by $s_n(D)$ the number of states of D consisting of n circles. The first aim of this paper is to characterize the virtual knot diagrams with $s_3(D) = 0$ in terms of their Gauss diagrams. The 3-state number of a virtual knot K is defined to be the minimal number of $s_3(D)$ among all possible diagrams D for K. The second aim of this paper is to study several properties of the virtual knots with $s_3(K) = 0$.

1. Introduction

A virtual knot diagram is a plane curve equipped with real crossings and virtual crossings. A virtual knot is an equivalence class of virtual knot diagrams under the generalized Reidemeister moves. In virtual knot theory introduced by L.H. Kauffman [2], the Jones polynomial is a state sum invariant defined by using a virtual knot diagram similarly to the classical case.

For a positive integer n, an *n-state* of a virtual knot diagram D is a union of n circles obtained from D by splicing all real crossings. Let $s_n(D)$ denote the number of n-states of D. In our previous paper [4], we

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studied several properties of $s_n(D)$ and give some lower and upper bounds for $s_1(D)$, $s_2(D)$, and $s_3(D)$ in particular. It is proved that $s_1(D) > 0$ for any diagram D, and that $s_2(D) = 0$ if and only if D has no real crossing.

On the other hand, there are infinitely many virtual knot diagrams D with $s_3(D) = 0$. Therefore, it is natural to consider the following question: *Which virtual knot diagram has no 3-state?* The first aim of this paper is to answer this question in terms of its Gauss diagram.

Theorem 1.1. *Let D be a virtual knot diagram with r real crossings, and G the Gauss diagram of D. Then the following are equivalent.*

(1) $s_3(D) = 0$.
(2) *The linking graph of G is linear.*
(3) $s_1(D) = \dfrac{2^{r+1} - (-1)^{r+1}}{3}$, $s_2(D) = \dfrac{2^r - (-1)^r}{3}$, *and* $s_n(D) = 0$ $(n \geq 3)$.

The definition of the linking graph of a Gauss diagram will be given in Section 3.

The *n-state number* of a virtual knot K is the minimal number of $s_n(D)$ for all possible virtual knot diagrams D of K, and denoted by $s_n(K)$ [4]. As an application of Theorem 1.1, we study several properties of a virtual knot K with $s_3(K) = 0$ by using three virtual knot invariants; the Miyazawa polynomial $R_K(A, \vec{x}) \in \mathbb{Z}[A, A^{-1}, x_1, x_2, \ldots]$, the n-writhe $J_n(K)$, and the upper/lower groups $G_+(K)$ and $G_-(K)$. Then we have the following.

Theorem 1.2. *Let K be a virtual knot with $s_3(K) = 0$. Then K satisfies one of the following conditions.*

(1) *The coefficient of x_1^2 in $R_K(A, \vec{x})$ is non-zero.*
(2) *Both $J_1(K)$ and $J_{-1}(K)$ are non-zero.*
(3) *$G_+(K)$ is not isomorphic to $G_-(K)$, or both $G_+(K)$ and $G_-(K)$ are isomorphic to $\mathbb{Z}$.*

As a corollary of Theorem 1.2, we obtain the following.

Corollary 1.3. *For any non-trivial classical knot K, we have $s_3(K) \geq 1$.*

This paper is organized as follows. In Section 2, we give basic definitions for virtual knots, virtual knot diagrams, and Gauss diagrams. In Section 3, we study several properties of the number of 3-states $s_3(D)$ and prove Theorem 1.1. In Section 4, we introduce three virtual knot invariants; the Miyazawa polynomial, the n-writhe, and the upper/lower groups. We prove Theorem 1.2 in Sections 5 and 6. We give some examples in Section 7.

2. Preliminaries

A μ-component *virtual link diagram* D is a union of immersed μ circles in $\mathbb{R}^2$ with transverse double points, which are classified into *real crossings* and *virtual crossings*. Each virtual crossing is assigned by a small circle, and two lines at each real crossing are distinguished as an over-crossing and an under-crossing such that the under-crossing line is deleted from the curve. If D has no virtual crossing, then D is called a *classical link diagram*. A 1-component link diagram is called a *knot* diagram simply.

A *virtual link* L is an equivalence class of virtual link diagrams under the seven kinds of Reidemeister moves as shown in Fig. 1. If L has a classical link diagram, then L is called a *classical link*. In this paper, we sometimes think L as an oriented link so that every real crossing admits the sign according to the orientations of over- and under-crossings as in the standard manner.