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Delta-crossing number for knots

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A Delta-crossing tangle is a tangle of three arcs with three crossings, which appears in a Delta move (or Delta unknotting operation). A Delta-crossing diagram is a diagram which can be decomposed into Delta-crossing tangles joined by simple arcs. We prove that every knot has a Delta-crossing diagram, and then investigate the Delta-crossing number which is the minimum number of Delta-crossing tangles among all Delta-crossing diagrams of the given knot. We obtain upper and lower bounds on the number in terms of the ordinal crossing number and genus. We also determine the number for prime knots with nine crossings or less except six knots.

1. Introduction

A knot diagram plays a fundamental role in the study of knots. The *crossing number* of a knot K, denoted by $c(K)$, is the minimum number of crossings among all diagrams. We consider a (small) disk-neighborhood of each crossing as a *single-crossing tangle*, which is a tangle of two arcs with a single crossing as in Fig. 1 (1). It can be easily seen that every knot diagram can be decomposed into single-crossing tangles joined by simple arcs. See Fig. 1 (2).

A *two-crossing tangle* is defined to be a tangle of two arcs with two crossings as in Fig. 2. A *two-crossing diagram* (or *matched diagram*) is a diagram which can be decomposed into two-crossing tangles joined by simple arcs. The question "Can a knot have a two-crossing diagram?" is close to Problem 1.60 proposed by Przytycki in Kirby Problem [6] "There are oriented knots without a matched diagram?" (cf. [10]). Duzhin [3] gives an example of a knot without a matched diagram.

A *three-crossing tangle* is defined to be a tangle of two arcs with three crossings as in Fig. 3. A *three-crossing diagram* is a diagram which can be decomposed into three-crossing tangles joined by simple arcs. The question "Can a knot have a three-crossing diagram?" is closely related to the 3-move conjecture

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(1) (2)

Fig. 1. A single-crossing tangle and a single-crossing diagram.

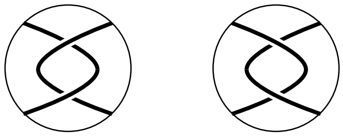


Fig. 2. Two-crossing tangles.

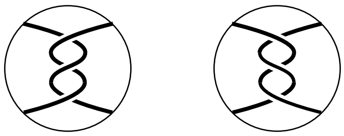
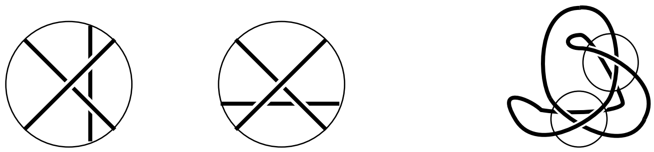


Fig. 3. Three-crossing tangles.



(1) (2)

Fig. 4. Triple-crossing tangles and a triple-crossing diagram.

(cf. Problem 1.59 in [6]), which is negatively answered by Dąbkowski and Przytycki [4]. Therefore, there is a knot without a three-crossing diagram.

A *triple-crossing tangle* (cf. [1]) is defined to be a tangle of three arcs with three crossings as in Fig. 4 (1), which appears in a third Reidemeister move. A *triple-crossing diagram* is a diagram which can be decomposed into triple-crossing tangles joined by simple arcs. See Fig. 4 (2)

Theorem 1. ([1]) *For any knot K, there exists a triple-crossing diagram of K.*

Adams also defines the *triple-crossing number* of a knot K, denoted by $c_3(K)$, by the minimum number of triple-crossing tangles among all triple-crossing diagrams of K.

Theorem 2. ([1]) $c(K)/3 \leq c_3(K) \leq c(K) - 1$. *(The second equality holds iff K is a $(2, p)$-torus knot.)*

In this paper, a Δ-*crossing tangle* is defined to be a tangle of three arcs with three crossings as in Fig. 5 (1), which are appeared in a Δ move (or Δ unknotting operation) [7]. A Δ-*crossing diagram* is a diagram which can be decomposed into Δ-crossing tangles joined by simple arcs. See Fig. 5 (2).

Theorem 3. *For any knot K, there exists a Δ-crossing diagram of K.*

We define the Δ-*crossing number* of a knot K, denoted by $c_\Delta(K)$, by the minimum number of Δ-crossing tangles among all Δ-crossing diagrams of K.