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Bridge genus and braid genus of lens space

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ABSTRACT

The bridge genus and the braid genus are invariants of a closed connected orientable 3-manifold introduced by A. Kawauchi. The Heegaard genus, the bridge genus and the braid genus are linearly independent as invariants of a closed connected orientable 3-manifold. In this paper, we calculate the bridge genus and the braid genus for some lens spaces, and we give the upper bounds of the braid genus for all lens spaces.

1. Introduction

Let M be a closed connected orientable 3-manifold, and L a link in the 3-sphere S^3. In the paper [3], A. Kawauchi introduced the bridge genus $g_{\text{bridge}}(M)$ and the braid genus $g_{\text{braid}}(M)$ for any M as follows. The *bridge genus* $g_{\text{bridge}}(M)$ (resp. the *braid genus* $g_{\text{braid}}(M)$) for any M is the minimal number of the bridge numbers bridge(L) (resp. the braid indexes braid(L)) for all links L such that M is obtained by the 0-surgery on S^3 along L. Here bridge(L) is the bridge number of L, and braid(L) is the braid index of L. Every closed connected orientable 3-manifold is obtained by the 0-surgery on S^3 along a link [3]. Thus, the bridge genus and the braid genus are invariants of M. Let $g_{\text{H}}(M)$ be the Heegaard genus for any M. The following theorem holds for $g_{\text{H}}(M)$, $g_{\text{bridge}}(M)$ and $g_{\text{braid}}(M)$ [5].

Theorem 1.1. *([5]) The following inequalities hold for every closed connected orientable 3-manifold M.*

$$g_{\text{H}}(M) \leq g_{\text{bridge}}(M) \leq g_{\text{braid}}(M).$$

Further, these invariants are linearly independent.

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For every lens space except the 3-sphere, the Heegaard genus is equal to one. In this paper, we consider the bridge genus and the braid genus of a lens space. Let $L(p,q)$ be the lens space of type (p,q), where we assume $0 < q < p$ unless otherwise specified.

In Section 2, we prove the following lemmas.

Lemma 1.2. *Let a be an even integer. Then the following equalities hold.*

$$g_{\text{bridge}}(L(a,1)) = g_{\text{braid}}(L(a,1)) = 3$$

Lemma 1.3. *Let a and b be even integers. Then the following equalities hold.*

$$g_{\text{bridge}}(L(ab-1,b)) = g_{\text{braid}}(L(ab-1,b)) = 4$$

Let $[a_1, a_2, \ldots, a_n]$ be the continued fraction of the quotient $\frac{p}{q}$ given as follows.

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}.$$

We define the non-negative integer $n(L(p,q))$ as follows.

$$n(L(p,q)) = \min\left\{ n;\ \begin{array}{l} [a_1, a_2, \ldots, a_n] = \frac{p'}{q'}, L(p',q') = L(p,q), \\ \text{and } a_1, a_2, \ldots, a_n \text{ are non-zero even integers} \end{array} \right\}.$$

In this definition, we grant that p' and q' are negative coprime integers. The following lemma gives the upper bound of the braid genus for all lens spaces.

Lemma 1.4. *For every lens space $L(p,q)$, the following inequality holds.*

$$g_{\text{braid}}(L(p,q)) \leq n(L(p,q)) + 2.$$

In Sections 3 and 4, we consider $g_{\text{bridge}}(L(p,q))$ and $g_{\text{braid}}(L(p,q))$ which are not obtained by Lemmas 1.2 and 1.3. By the following theorem, we have examples of a lens space whose bridge genus and braid genus are 5.

Theorem 1.5. *For the lens space $L(p,q)$, if $n(L(p,q)) = 3$, and p is an even integer such that $\frac{p}{2}$ is a prime integer or the product of two (possibly, equal) prime integers, and q is not a square modulo p, then*

$$g_{\text{bridge}}(L(p,q)) = g_{\text{braid}}(L(p,q)) = 5.$$

We have the following example by Theorem 1.5.

Example 1.6. *The following equalities hold.*

$$g_{\text{bridge}}(L(8,3)) = g_{\text{braid}}(L(8,3)) = 5.$$
$$g_{\text{bridge}}(L(10,3)) = g_{\text{braid}}(L(10,3)) = 5.$$