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The hyperspace of convergent sequences ☆

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We dedicate this paper to Prof. T. Nogura on his retirement

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A B S T R A C T

The hyperspace of nontrivial convergent sequences of a metric space X without isolated points will be denoted by $\mathcal{S}_c(X)$. This hyperspace is equipped with the Vietoris Topology. It is not hard to prove that $\mathcal{S}_c([0,1])$ and $\mathcal{S}_c(\mathbb{I})$ are not homeomorphic, where $\mathbb{I}$ are the irrationals. We show that the hyperspaces $\mathcal{S}_c(\mathbb{R})$ and $\mathcal{S}_c([0,1])$ are path-wise connected. In a more general context, we show that if X is path-wise connected space, then $\mathcal{S}_c(X)$ is connected. But $\mathcal{S}_c(X)$ is not necessarily path-wise connected even when X is the Warsaw circle. These make interesting to study the connectedness of the hyperspace of nontrivial convergent sequences in the realm of continua. Also, we prove that if X is a second countable space, then $\mathcal{S}_c(X)$ is meager. We list several open questions concerning this hyperspace.

1. Preliminaries and introduction

The Greek letter ω stands for the first infinite cardinal number and ω_1 stands for the first uncountable cardinal number. The symbol $[0, \omega_1)$ will denote the ordinal space with the order topology. The letters $\mathbb{R}$, $\mathbb{I}$, $\mathbb{Q}$ and $\mathbb{N}$ will denote the real numbers, the irrational numbers, the rational numbers and the natural numbers, respectively. For our convenience, sometimes we shall assume that $0 \notin \mathbb{N}$.

The hyperspaces have been extensively studied in topology. Most of the published articles about hyperspaces have been concentrated around the families of closed subsets, compact subsets, subcontinua and finite subsets equipped with several topologies. The Vietoris topology is the most recurrent topology for the hyperspaces:

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For a topological space $(X, \mathcal{T})$, $\mathcal{CL}(X)$ will denote the set of all nonempty closed subsets of X and $\mathcal{K}(X)$ the set of nonempty compact subsets of X. For each nonempty subset V of X we define

$$V^+ = \{A \in \mathcal{CL}(X) : A \subseteq V\} \quad \text{and} \quad V^- = \{A \in \mathcal{CL}(X) : A \cap V \neq \emptyset\}.$$

The *Vietoris topology* is the topology on $\mathcal{CL}(X)$ generated by the sets of the form U^+ and U^-, where U is a nonempty open subset of X. A basic open set of the Vietoris topology is of the form

$$\langle U_1, \ldots, U_n \rangle = \Big(\bigcup_{i \le n} U_k\Big)^+ \cap \Big(\bigcap_{i \le n} U_i^-\Big),$$

where $U_1, \ldots, U_n$ are nonempty open subsets of X. For each space X, the hyperspace $\mathcal{K}(X)$ will be considered as a subspace of $\mathcal{CL}(X)$. Those notions used and not defined in this article have the meaning given to them in [3].

Concerning metric spaces without isolated points (crowded), it is well-known that several topological properties of a metric space are determined by its nontrivial convergent sequences. This makes very natural to ask questions about the topological properties of the hyperspace of nontrivial convergent sequences of a metric space, mainly when the space does not have isolated points like continua. Although, some results will be stated for arbitrary topological spaces, it is going to be clear, in the context, when the involved space will be metric without isolated points.

To begin the study of the hyperspace of nontrivial convergent sequences we need to introduce some terminology. First, a nontrivial convergent sequence will be an infinite countable set $S \in \mathcal{CL}(X)$ for which there exists $x \in S$ such that $S \setminus \{x\}$ is discrete and $|S \setminus U| < \omega$ for every neighborhood U of x. The point x is called the *limit point* of S and we say that S converges to x (in symbols, $S \to x$). In the standard notation, if $(x_n)_{n \in \mathbb{N}}$ denotes a nontrivial sequence of a space X converging to x, then we shall write $\{x, x_n : n \in \mathbb{N}\}$ to refer to the sequence and its limit point. In this notation $\{x, x_n : n \in \mathbb{N}\}$ we shall understand, in most of the cases, that $x \neq x_n$ for every $n \in \mathbb{N}$ and $x_n \neq x_m$ for every pair of distinct natural numbers $n \neq m$. The hyperspace of nontrivial convergent sequences of X with the Vietoris topology will be denoted by $\mathcal{S}_c(X)$.

In general, the hyperspace $\mathcal{CL}(X)$ fails to be normal. Indeed, $\mathcal{CL}(X)$ is normal iff X is compact (for a proof of this fact see [6]). On the other hand, for the hyperspace of compact nonempty subsets, E. Michael [5] proved that a space X is metrizable if and only if the hyperspace $\mathcal{K}(X)$ is metrizable. In particular, $\mathcal{CL}(\mathbb{R})$ cannot be metrizable and $\mathcal{K}(\mathbb{R})$ is metrizable. Since $\mathcal{S}_c(X)$ is considered as a subspace of $\mathcal{K}(X)$, $\mathcal{S}_c(X)$ is metrizable whenever X is metrizable. Moreover, if X is first countably, then $\mathcal{S}_c(X)$ is dense in $\mathcal{CL}(X)$. The first natural questions concerning the hyperspace of nontrivial convergent sequences that can be formulated are the following.

(1) Are $\mathcal{S}_c([0,1])$ and $\mathcal{S}_c([0,\omega_1))$ homeomorphic?
(2) Are $\mathcal{S}_c([0,1])$ and $\mathcal{S}_c(\mathbb{I})$ homeomorphic?
(3) Are $\mathcal{S}_c(\mathbb{I})$ and $\mathcal{S}_c([0,\omega_1))$ homeomorphic?

These questions can be solved in a negative way by using the next lemma:

Lemma 1.1. *Let X be a topological space and let $A \subseteq X$. Then, for every nontrivial convergent sequence $S \subseteq X \setminus A$, the function $f : A \to \mathcal{S}_c(X)$ defined by $f(x) = S \cup \{x\}$ for each $x \in A$ is an embedding from A into $\mathcal{S}_c(X)$.*

Proof. It is clear that the function f is well defined and injective. Let $x \in A$ and let $U_1, \ldots U_n$ be pairwise disjoint open subsets of X such that $S \cup \{x\} \in \mathcal{U} = \langle U_1, \ldots, U_n \rangle$. Pick $y \in f^{-1}[\mathcal{U}]$ and pick $k \le n$ so that $y \in U_k$. Then, we have that $y \in U_k \cap A$, which is a nonempty open subset of A, and $f[U_k \cap A] \subseteq \mathcal{U}$.