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Essential tangle decompositions of knots with tunnel number one tangles

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ABSTRACT

It is shown by Ozawa that a knot in the 3-sphere has a unique essential tangle decomposition if it admits an essential free 2-tangle decomposition. We show that Ozawa's result cannot be generalized even if a knot admits an essential 2-tangle decomposition such that one of the decomposed tangles is of tunnel number one.

1. Introduction

For a positive integer m, an *m-tangle* (B, T) is defined to be a pair of a 3-ball B and mutually disjoint m arcs T properly embedded in B. Let K be a knot in the 3-sphere S^3 and $P \subset S^3$ a 2-sphere intersecting K in $2m$ points. Then P cuts S^3 into two 3-balls, say B_1 and B_2. Since P intersects K in $2m$ points, we see that each B_i $(i = 1, 2)$ intersects K in a collection of mutually disjoint m arcs, say T_i. Hence each (B_i, T_i) is an m-tangle. The decomposition $(B_1, T_1) \cup_{\mathcal{P}} (B_2, T_2)$ is called an *m-tangle decomposition* of (S^3, K), where $\mathcal{P} = (P, P \cap K)$. We call $\mathcal{P}$ a *tangle sphere* or an *m-tangle sphere*. A tangle decomposition $(B_1, T_1) \cup_{\mathcal{P}} (B_2, T_2)$ is said to be *essential* if $\mathcal{P}$ is incompressible in (S^3, K).

In previous work [3], the author defined *tunnel number*, denoted by $\mathrm{tnl}(\cdot)$, of a tangle which is a natural generalization of tunnel number of a knot. See the next section for definitions. We here notice that a tangle (B, T) is of tunnel number zero if and only if it is a *free* tangle, i.e., the exterior of T in B is homeomorphic to a handlebody. A tangle decomposition $(B_1, T_1) \cup_{\mathcal{P}} (B_2, T_2)$ is said to be *free* if each (B_i, T_i) is a free tangle.

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Theorem 1.1. *(Ozawa [2, Theorem 1.2]) Let K be a knot in S^3. Suppose that $\mathcal{P}$ gives an essential free 2-tangle decomposition. Then any tangle sphere giving an essential tangle decomposition of (S^3, K) is ambient isotopic to $\mathcal{P}$.*

The following is merely a restatement of Theorem 1.1 by using the notation $\mathrm{tnl}(\cdot)$.

Theorem 1.2 *(Restatement of Theorem 1.1). Let K be a knot in S^3 with an essential 2-tangle decomposition $(B_1, T_1) \cup_{\mathcal{P}} (B_2, T_2)$ with $\mathrm{tnl}(T_i) = 0$ for each $i = 1, 2$. Then any tangle sphere giving an essential tangle decomposition of (S^3, K) is ambient isotopic to $\mathcal{P}$.*

In this paper, we show that Theorem 1.2 cannot be generalized even if a knot admits an essential 2-tangle decomposition such that one of the decomposed tangles is of tunnel number one.

Theorem 1.3. *For any non-negative integer n, there is a knot K in S^3 which satisfies the following:*

(1) *(S^3, K) admits an essential 2-tangle decomposition $(B_1, T_1) \cup_{\mathcal{P}} (B_2, T_2)$ with $\mathrm{tnl}(T_i) = 1$ and $\mathrm{tnl}(T_j) = n$ for $(i, j) = (1, 2)$ or $(2, 1)$.*
(2) *(S^3, K) admits another essential 2-tangle decomposition different from the above.*

2. Definitions

Throughout this paper, we work in the piecewise linear category. Let B be a sub-manifold of a manifold A. The notation $\mathrm{Nbd}(B; A)$ denotes a (closed) regular neighborhood of B in A. By $\mathrm{Ext}(B; A)$, we mean the *exterior* of B in A, i.e., $\mathrm{Ext}(B; A) = \mathrm{cl}(A \setminus \mathrm{Nbd}(B; A))$, where $\mathrm{cl}(\cdot)$ means the closure. The notation $|\cdot|$ indicates the number of connected components. Let M be a compact connected orientable 3-manifold with non-empty boundary. Let J be a 1-manifold properly embedded in M and F a surface properly embedded in M. Here, a *surface* means a connected compact 2-manifold. We always assume that a surface intersects J transversely. Set $\mathcal{M} = (M, J)$ and $\mathcal{F} = (F, F \cap J)$. For convenience, we also call $\mathcal{F}$ a *surface*. A simple closed curve properly embedded in $F \setminus J$ is said to be *inessential* in $\mathcal{F}$ if it bounds a disk in F intersecting J in at most one point. A simple closed curve properly embedded in $F \setminus J$ is said to be *essential* in $\mathcal{F}$ if it is not inessential in $\mathcal{F}$. A surface $\mathcal{F}$ is *compressible* in $\mathcal{M}$ if there is a disk $D \subset M \setminus J$ such that $D \cap F = \partial D$ and ∂D is essential in $\mathcal{F}$. Such a disk D is called a *compressing disk* of $\mathcal{F}$. We say that $\mathcal{F}$ is *incompressible* in $\mathcal{M}$ if $\mathcal{F}$ is not compressible in $\mathcal{M}$.

A 3-manifold C is called a (genus g) *compression body* if there exists a closed surface F of genus g such that C is obtained from $F \times [0, 1]$ by attaching 2-handles along mutually disjoint loops in $F \times \{0\}$ and filling in some resulting 2-sphere boundary components with 3-handles. We denote $F \times \{1\}$ by $\partial_+ C$ and $\partial C \setminus \partial_+ C$ by $\partial_- C$. A compression body C is called a *handlebody* if $\partial_- C = \emptyset$. The triplet $(C_1, C_2; S)$ is called a (genus g) *Heegaard splitting* of M if C_1 and C_2 are (genus g) compression bodies with $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = S$. The Heegaard genus $\mathrm{hg}(M)$ of M is the minimal integer g for which M admits a genus g Heegaard splitting.

A simple arc γ properly embedded in a compression body C is said to be *vertical* if γ is isotopic to an arc with $\{\text{a point}\} \times [0, 1] \subset \partial_- C \times [0, 1]$ relative to boundary. A simple arc γ properly embedded in C is said to be *trivial* if there is a disk δ in C with $\gamma \subset \partial\delta$ and $\partial\delta \setminus \gamma \subset \partial_+ C$. Such a disk δ is called a *bridge disk* of γ. A disjoint union of trivial arcs is said to be *mutually trivial* if they admit a disjoint union of bridge disks.

We now recall definitions of a *c-compression body* and a *c-Heegaard splitting* given by Tomova [4]. Let J be a 1-manifold properly embedded in a compact connected orientable 3-manifold M with non-empty boundary. A surface $\mathcal{F} = (F, F \cap J)$ is *c-compressible in* $\mathcal{M} = (M, J)$ if there is a disk $D \subset M \setminus J$ such