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Two definitions of the bridge index of a welded knot

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ABSTRACT

There are two equivalent definitions of the bridge index of classical knots; one is the minimal number of the over-bridges, and the other is that of the maximal points with respect to a hight function. We consider these two indices for virtual and welded knots, and prove an inequality for virtual knots and the equality for welded knots.

1. Introduction

In classical knot theory, there are two equivalent definitions of the bridge index of a classical knot K: First, for a classical diagram D of K illustrated in the plane, let $b_1(D)$ be the number of over-bridges which pass all over-crossings but no under-crossings, and $\mathrm{b}_1(K)$ be the minimal number of $b_1(D)$ for all classical diagrams D of K. Second, let $b_2(D)$ be the number of maximal points on D with respect to a fixed axis of the plane, and $\mathrm{b}_2(K)$ the minimal number of $b_2(D)$ for all classical diagrams D of K. Then it holds that $\mathrm{b}_1(K) = \mathrm{b}_2(K)$, which is called the *bridge index* of a classical knot K.

In virtual knot theory [4], these two numbers $\mathrm{b}_1(K)$ and $\mathrm{b}_2(K)$ for a virtual knot K do not coincide generally. Here, an over-bridge is an arc on a virtual knot diagram which passes over-crossings and/or virtual crossings but no under-crossings. In fact, there are infinitely many virtual knots with $\mathrm{b}_1(K) = 1$, whereas $\mathrm{b}_2(K) = 1$ implies that K is the trivial knot [2]. The first aim of this paper is to prove the following.

Theorem 1.1. *For any virtual knot K, it holds that $\mathrm{b}_1(K) \leq \mathrm{b}_2(K)$.*

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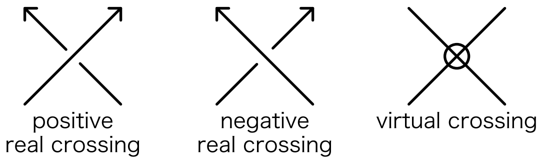

Fig. 1. Three types of crossings.

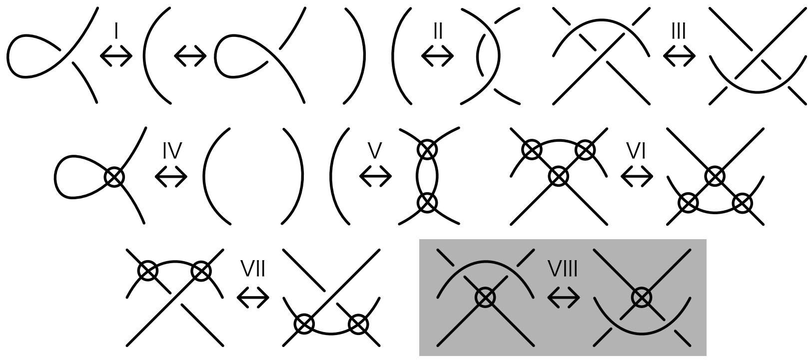

Fig. 2. Eight kinds of local moves.

A welded knot is an equivalence class of virtual knots under the "welded" move where an arc goes over a virtual crossing introduced in [1]. The second aim of this paper is to study the two numbers $\mathrm{b}_1(K)$ and $\mathrm{b}_2(K)$ in welded knot theory.

Theorem 1.2. *For any welded knot K, it holds that $\mathrm{b}_1(K) = \mathrm{b}_2(K)$. Furthermore, there is a virtual knot diagram D of K which realizes the two minimal numbers simultaneously.*

This paper is organized as follows. In Section 2, we review the definitions of virtual knot diagrams, Gauss diagrams, and two kinds of bridge indices. In Section 3, we prove Theorems 1.1 and 1.2.

2. Definitions

A *virtual knot diagram* D is a circle immersed in the plane $\mathbb{R}^2$ such that the multiple points are transverse double points classified into two types called *real crossings* and *virtual crossings* [4]. At a real crossing, there are two intersecting paths, *over-path* and *under-path*, distinguished by cutting the under-path into two pieces. A virtual crossing is surrounded by a small circle. Throughout this paper, all virtual knot diagrams are assumed to be oriented. Each real crossing is equipped with a positive or negative sign in a usual way. See Fig. 1.

We consider eight kinds of local moves I–VIII for virtual knot diagrams as shown in Fig. 2. A *virtual knot* is an equivalence class of virtual knot diagrams under the moves I–VII, and a *welded knot* is an equivalence class under I–VII and VIII [1]. We remark that it is forbidden to use a move similar to VIII with opposite crossing information: In fact, if we are allowed to use such moves together, then any virtual knot diagram is equivalent to the one with no crossing [3,5].

Let D be a virtual knot diagram with n real crossings $c_1, \ldots, c_n$, and $f : S^1 \to \mathbb{R}^2$ an immersion of a circle S^1 into $\mathbb{R}^2$ such that the image $f(S^1)$ presents the underlying curve of D. For each real crossing c_i, let a_i and b_i be the preimage of c_i such that a small neighborhood of a_i (or b_i, resp.) is mapped into the over-path (or under-path) at c_i, respectively. The point a_i (or b_i) is called the *over-crossing* (or *under-crossing*) of c_i. The *Gauss diagram* of D is the union of the circle S^1 and n chords $\gamma_1, \ldots, \gamma_n$ such that γ_i connects