Topology and its Applications 196 (2015) 904–910

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# Examples of hyperpolar actions of the automorphism groups of Lie algebras

Yuichiro Taketomi

*1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526, Japan*

ARTICLE INFO

*Article history:*
Received 6 January 2014
Received in revised form 4 February 2014
Accepted 5 February 2014
Available online 28 May 2015

*MSC:*
53C35
57S20
53C40

*Keywords:*
Hyperpolar action
Noncompact symmetric space

ABSTRACT

We are studying isometric actions obtained by the automorphism groups of Lie algebras on the noncompact symmetric space $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$. In this paper, we show that the actions which come from some solvable Lie algebras provide examples of hyperpolar actions with singular orbit and of higher cohomogeneity.



## 1. Introduction

Isometric actions on Riemannian symmetric spaces provide many interesting examples of isometric actions, and have been studied actively by many authors, for examples, in [4,5,8]. Recently, left-invariant metrics on a Lie group have been studied from the viewpoint of isometric actions on the noncompact symmetric space $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$. Let $G$ be a simply connected Lie group with the Lie algebra $\mathfrak{g}$. Let us put

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) := \{c\varphi \in \mathrm{GL}(\mathfrak{g}) \mid c \in \mathbb{R}^\times,\ \varphi[\cdot,\cdot] = [\varphi\cdot, \varphi\cdot]\}.$$

Kodama, Takahara and Tamaru [7] have pointed out that the orbit space of the action of $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})$ on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ can be identified with a kind of moduli space of left-invariant metrics on $G$.

An orbit of a group action on a manifold is called regular if its dimension is maximal. An orbit which is not regular is called singular. A group action on a manifold is said to be of cohomogeneity $k$ if a regular orbit of the action is of codimension $k$. Some examples of cohomogeneity one $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})$-actions have been

*E-mail address:* y-taketomi@hiroshima-u.ac.jp.

http://dx.doi.org/10.1016/j.topol.2015.05.057
0166-8641/

found. Hashinaga and Tamaru [6] have proved that if $\mathfrak{g}$ is a three-dimensional solvable Lie algebra then the $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-action is transitive or of cohomogeneity one. The author and Tamaru have constructed some examples of cohomogeneity one $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-actions for higher-dimensional $\mathfrak{g}$. They have proved:

**Proposition 1.1.** *([10]) For every $n \geq 3$, the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}^n_{1,1})$ on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ is of cohomogeneity one. Here, $\mathfrak{g}^n_{1,1}$ is an $n$-dimensional Lie algebra with a basis $\{e_1, \ldots, e_n\}$ whose bracket relations are given by*

$$\begin{cases} [e_1, e_n] = e_1, \\ \vdots \\ [e_{n-2}, e_n] = e_{n-2}, \\ [e_{n-1}, e_n] = e_1 + e_{n-1}. \end{cases}$$

Other examples of cohomogeneity one $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-actions have been constructed by Kodama, Takahara and Tamaru [7]. Hence, one might expect that the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ provides interesting examples of isometric actions.

Let $n \in \mathbb{N}$ with $n \geq 2$, and $k \in \mathbb{N} \cup \{0\}$ with $k < n/2$. We define $\mathfrak{s}^n_k$ as an $n$-dimensional Lie algebra with a basis $\{e_1, \ldots, e_n\}$ whose bracket relations are given by

$$\begin{cases} [e_1, e_n] = e_1, \\ \vdots \\ [e_{n-k-1}, e_n] = e_{n-k-1}, \\ [e_{n-k}, e_n] = e_1 + e_{n-k}, \\ \vdots \\ [e_{n-1}, e_n] = e_k + e_{n-1}. \end{cases}$$

We note that $\mathfrak{s}^n_1 = \mathfrak{g}^n_{1,1}$. Hence, $\mathfrak{s}^n_k$ is a kind of generalization of $\mathfrak{g}^n_{1,1}$. In this paper, we study $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{s}^n_k)$-actions. Recall that an isometric action of a connected Lie group $G$ on a Riemannian manifold $M$ is called hyperpolar if there exists a connected closed flat submanifold $\Sigma$ of $M$ such that $\Sigma$ meets each $G$-orbit and $\Sigma$ intersects each $G$-orbit orthogonally.

As our main result, we show that the actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{s}^n_k)$ provide new examples of higher cohomogeneity hyperpolar actions with singular orbits:

**Theorem 1.2.** *The actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{s}^n_k)$ on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ are hyperpolar actions of cohomogeneity $k$. Furthermore, the actions have singular orbits if $k \geq 2$.*

Hyperpolar actions on symmetric spaces have been of interest to many authors. Hyperpolar actions on simply connected irreducible symmetric spaces of compact type have been classified completely by Kollross [8]. On the other hand, the case of noncompact type seems to be far from complete understanding. We note that any cohomogeneity one action is hyperpolar. Cohomogeneity one actions on irreducible symmetric spaces of noncompact type have been studied deeply by Berndt and Tamaru [5], and well-understood. In higher cohomogeneity cases, hyperpolar actions on irreducible symmetric spaces of noncompact type without singular orbit have been classified completely by Berndt, Díaz-Ramos, and Tamaru [4]. On the other hand, little is known about hyperpolar action with singular orbits on symmetric spaces of noncompact type [1].

We note that if $k < 2$ then the actions have no singular orbits. In fact, $\mathfrak{s}^n_0$ coincides with the Lie algebra of real hyperbolic space, and the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{s}^n_0)$ is transitive [7,9]. In the case of $k = 1$, it has been proved that all $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{s}^n_1)$-orbits are congruent to each other [10].

Our work is organized as follows. We determine $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{s}^n_k)$ in Section 2. In Section 3, we review some basic facts from Riemannian geometry, and prove Theorem 1.2.