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On slice knots and smooth 4-manifolds

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ABSTRACT

As a consequence of Donaldson's result, concerning the intersection forms of smooth 4-manifolds, we show that there exists an exotic smooth structure on $\sharp\infty\mathbb{CP}^2$.

1. Introduction

For $n \neq 4$, the n-dimensional Euclidean space $\mathbb{R}^n$ admits a unique smooth structure. However there exist uncountably many smooth structures on $\mathbb{R}^4$ [7]. Moreover, every punctured, smooth, closed 4-manifold has uncountably many distinct smooth structures [6,9]. This emphasizes the special behavior of 4-manifolds.

We say that a smooth 4-manifold X has an *exotic smooth structure* if there exists a smooth 4-manifold X' which is homeomorphic to X, but not diffeomorphic to X. We consider the following problem.

Problem. Does there exist an exotic smooth structure on a given 4-manifold?

In recent years there have been numerous works on exotic smooth structures on a closed 4-manifold which is homeomorphic to $\sharp m\mathbb{CP}^2\sharp n\overline{\mathbb{CP}^2}$ where m and n are finite. (Here $\sharp m\mathbb{CP}^2\sharp n\overline{\mathbb{CP}^2}$ means the connected sum of $m\mathbb{CP}^2$'s and $n\overline{\mathbb{CP}^2}$'s.)

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Fang showed that if M is a closed 3-dimensional topological flat submanifold of $\sharp n\overline{\mathbb{CP}^2}$, then there exist uncountably many smooth structures on $M \times \mathbb{R}$ which are all smoothly embedded into $\sharp n\overline{\mathbb{CP}^2}$ [3]. Then this result was generalized for any open 4-manifold, topologically embedded in $\sharp n\mathbb{CP}^2$ [10, p. 383].

Let M be an oriented, connected, simply connected, smooth 4-manifold. We define the connected sum of infinitely many M's, $\sharp\infty M$, as the universal covering space of $T^4\sharp M$ where T^4 is the 4-torus. In this paper, we show the following.

Theorem 1.1. *Any non-compact, connected, oriented, smooth 4-submanifold of $\sharp\infty\mathbb{CP}^2$ admits an exotic smooth structure.*

Throughout this paper, a manifold is always assumed to be oriented. This paper is organized as follows. In Section 2, we shall define a *z-exotic knot* in a smooth 4-manifold and show that any non-compact, connected, smooth 4-manifold with such a knot, admits an exotic smooth structure. In Section 3, we prove Theorem 1.1.

2. Smooth structures on non-compact 4-manifolds

Let M and N be smooth manifolds. For a positive integer n, we denote by $M\sharp N$ the connected sum of M and N and by $\sharp nM$ the connected sum of n copies of the same manifold M. We denote by $\operatorname{Int} M$ the interior of M and by ∂M the boundary of M.

Definition 2.1. Let V be a smooth and oriented 4-manifold. If there is a 4-ball B^4 smoothly embedded in V such that ∂B^4 contains a knot K and there is a 2-disk smoothly (resp. topologically locally flatly) properly embedded in $V \setminus \operatorname{Int} B^4$ such that the boundary of the disk is K and the disk is trivial in $H_2(V \setminus \operatorname{Int} B^4, \partial(V \setminus \operatorname{Int} B^4); \mathbb{Z})$ (that is the disk is *null-homologous*), then we say that K is *slice* (resp. *topologically slice*) in V. We call the embedded disk a *slice disk* (resp. a *topological slice disk*).

Clearly, if a knot is slice in V, then the knot is also topologically slice in V.

Definition 2.2. We say that a knot K is *z-exotic* in V if V is a smooth and oriented 4-manifold and K is topologically slice in V, but not slice in V.

The term “z-exotic” comes from the coefficient $\mathbb{Z}$ of the homology in Definition 2.1.

Proposition 2.3. *Any non-compact, connected, oriented, smooth 4-manifold which has a z-exotic knot K admits at least two smooth structures.*

Proof. Let M be a non-compact, connected, oriented, smooth 4-manifold. If M admits a z-exotic knot K, then there exists a smooth embedded 4-ball B^4 and a topological slice disk D for $K \subset \partial B^4$ with a topological embedding $\lambda : D^2 \times D^2 \to M \setminus \operatorname{Int} B^4$ ($\lambda(D^2 \times \{0\}) = D$) such that D represents zero in $H_2(M \setminus \operatorname{Int} B^4, \partial(M \setminus \operatorname{Int} B^4); \mathbb{Z})$. The (standard) smooth structures on $D^2 \times D^2$ and the 4-ball induce a smooth structure on $\lambda(D^2 \times D^2) \cup B^4$ by gluing them along the intersection. (Here we use uniqueness of smoothings on 3-manifolds.) We denote the resultant smooth 4-manifold by N_0 as in Fig. 1.

Note that there exists a smooth structure on the topological manifold $M \setminus \operatorname{Int}(\lambda(D^2 \times D^2) \cup B^4)$ because it is non-compact and connected (cf. [10, p. 377] or [5, §8]). We denote the smooth manifold by N. Now gluing N_0 and N along their boundaries, we have a new smooth 4-manifold, denoted by $\overline{M}$, such that $\overline{M}$ is homeomorphic to M because the gluing homeomorphism is isotopic to diffeomorphism by uniqueness of smoothings on 3-manifolds [10, p. 522]. If $\overline{M}$ were diffeomorphic to M, then K could be slice in M. This contradicts to the assumption. □