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# Separation of diagonal in monotonically normal spaces and their products ☆

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A B S T R A C T

As separation of diagonal, we study when monotone normality implies $\Delta$-paracompactness or $\Delta$-normality. For that, it is proved that every monotonically normal space is $\Delta$-paracompact if the projection of its square is closed. Moreover, it is proved that every monotonically normal space is $\Delta$-normal if it has countable tightness (or countable extent). In particular, the parenthetic part is an affirmative answer to Burke and Buzyakova's problem in 2010. Secondly, we study the relation between normality and $\Delta$-paracompactness or $\Delta$-normality in certain products. For that, we additionally introduce two new neighborhood properties. Using these ones, it is proved that the product $X \times K$ of a monotonically normal space $X$ and a compact space $K$ is $\Delta$-paracompact (respectively, $\Delta$-normal) if and only if $X$ is $\Delta$-paracompact (respectively, $\Delta$-normal) and $X \times K$ is normal.
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**Contents**



☆ Dedicated to the memory of Professor Ratislav Telgársky.

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## 1. Introduction

Throughout this paper, all spaces are assumed to be *Hausdorff*. For a space $X$, the diagonal $\{\langle x, x\rangle : x \in X\}$ of $X$ is denoted by $\Delta_X$. Note that the diagonal $\Delta_X$ of $X$ is closed in $X \times X$.

Buzyakova [3] gave the unified names for several diagonal separation properties, and studied their implications. Here, we take up the two properties of $\Delta$-paracompactness and $\Delta$-normality from these properties, because they seem to be essential.

**Definition 1.** A space $X$ is $\Delta$-*paracompact* if for each closed set $C$ in $X \times X$ disjoint from $\Delta_X$, there is a locally finite open cover $\mathcal{U}$ of $X$ such that $\bigcup\{U \times U : U \in \mathcal{U}\}$ misses $C$.

**Definition 2.** A space $X$ is $\Delta$-*normal* if for each closed set $C$ in $X \times X$ disjoint from $\Delta_X$, there are disjoint open sets $U$ and $V$ in $X \times X$ such that $C \subset U$ and $\Delta_X \subset V$.

As pointed out in [3,13], in the class of normal spaces, $\Delta$-paracompactness implies $\Delta$-normality and coincides with functional $\Delta$-paracompactness, which was known from long time ago and was called divisibility or strong collectionwise normality in other words (see [5]).

Buzyakova [3] proved that every GO-space is $\Delta$-paracompact. Immediately after, Burke and Buzyakova [2] proved that first countable, countably compact and monotonically normal space is $\Delta$-paracompact.

The concept of $\Delta$-normality was named and studied by Hart [7], who showed by a simple example that there is a monotonically normal space which is not $\Delta$-normal. As long as seeing these results, one may consider there is a big difference between monotone normality and these diagonal separation properties. However, to find a nice class of spaces for these diagonal separation properties, it is natural to consider the following problem.

**Problem A.** When is a monotonically normal space $\Delta$-paracompact or $\Delta$-normal?

In Section 2, as the preparation of proofs later, we state some notation and results for monotonically normal spaces. By Hart's example stated above, Problem A is negative without any additional condition. In Section 3, answering Problem A for $\Delta$-paracompactness, it is proved that every monotonically normal space is $\Delta$-paracompact if the projection of its square is closed. In Section 4, answering Problem A for $\Delta$-normality, it is proved that every monotonically normal space is $\Delta$-normal if it has countable tightness (or countable extent). This parenthetic part is an affirmative answer to [2, Problem 2.2].

Hart [7] also showed that the non-normal product $\omega_1 \times (\omega_1 + 1)$ is not $\Delta$-normal. On the other hand, normality of the products of a monotonically normal space and a compact space was characterized by some neighborhood properties in [9]. So we consider the following natural problems.

**Problem B.** Let $X$ be a $\Delta$-paracompact and monotonically normal space and $K$ a compact space.

(1) If $X \times K$ is $\Delta$-paracompact or $\Delta$-normal, is it normal?
(2) If $X \times K$ is normal, is it $\Delta$-paracompact or $\Delta$-normal?

In Sections 5 and 6, we introduce two new neighborhood properties and give affirmative answers to Problem B(1) for $\Delta$-paracompactness and $\Delta$-normality, respectively. In Sections 7 and 8, we also give affirmative answers to Problem B(2) for $\Delta$-paracompactness and $\Delta$-normality, respectively. Consequently, we can obtain two characterizations for $\Delta$-paracompactness and $\Delta$-normality of such products in terms of normality. In

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