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Regular versus continuous rational maps

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ABSTRACT

We prove that a continuous map from a compact nonsingular real algebraic variety X into the unit 2-sphere can be approximated by regular maps if and only if it is homotopic to a continuous map which is regular in the complement of a Zariski closed subvariety A of X of codimension at least 3. The assumption on the codimension of A is essential.

1. Introduction

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [2]). The class of real algebraic varieties is identical with the class of quasi-projective real varieties, cf. [2, Proposition 3.2.10, Theorem 3.4.4]. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let X and Y be real algebraic varieties. A map $f: X \to Y$ is said to be *continuous rational* if it is continuous and there exists a Zariski open and dense subvariety U of X such that the restriction $f|_U : U \to Y$ is a regular map. Let $U(f)$ denote the union of all such U. The complement $P(f) = X \setminus U(f)$ of $U(f)$ is called the *irregularity locus* of f. Thus $P(f)$ is the smallest Zariski closed subvariety of X for which the restriction $f|_{X\setminus P(f)} : X \setminus P(f) \to Y$ is a regular map. Continuous rational maps form a natural intermediate class between regular and continuous maps, with many specific properties, cf. [8,12–15].

In [13–15] investigated are continuous rational maps with values in the unit p-sphere

$$\mathbb{S}^p = \left\{(u_1, \dots, u_{p+1}) \in \mathbb{R}^{p+1} \;\middle|\; u_1^2 + \dots + u_{p+1}^2 = 1\right\}.$$

In certain cases, involving approximation or homotopy, such maps behave like regular maps. Let $\mathcal{C}(X, \mathbb{S}^p)$ denote the space of all continuous maps from X into $\mathbb{S}^p$, endowed with the compact-open topology. By definition, a continuous map $h: X \to \mathbb{S}^p$

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can be approximated by regular (resp. continuous rational) maps if every neighborhood of h in $\mathcal{C}(X, \mathbb{S}^p)$ contains a regular (resp. continuous rational) map.

The following fact is proved in [13, Corollary 1.4].

Proposition 1.1. *Let X be a compact nonsingular real algebraic variety. For a continuous map $h : X \to \mathbb{S}^1$, the following conditions are equivalent*:

(a) *h can be approximated by regular maps.*
(b) *h is homotopic to a regular map.*
(c) *h can be approximated by continuous rational maps.*
(d) *h is homotopic to a continuous rational map.*

It is well known that in general a continuous rational map from X into $\mathbb{S}^1$ cannot be approximated by regular maps, cf. [4] or [2, pp. 354–356].

In the present paper maps into even-dimensional spheres are considered. Proofs are postponed until Section 2.

Theorem 1.2. *Let X be a compact nonsingular real algebraic variety. For a continuous map $h : X \to \mathbb{S}^2$, the following conditions are equivalent*:

(a) *h can be approximated by regular maps.*
(b) *h is homotopic to a regular map.*
(c) *h can be approximated by continuous rational maps with irregularity locus of dimension at most* $\dim X - 3$.
(d) *h is homotopic to a continuous rational map with irregularity locus of dimension at most* $\dim X - 3$.

Approximation of continuous maps with values in $\mathbb{S}^2$ by regular maps is a very subtle problem, cf. [2,3,5,6].

For any positive integer p, let σ_p be a generator of the cohomology group $H^p(\mathbb{S}^p; \mathbb{Z}) \cong \mathbb{Z}$. Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ be the n-fold product.

Theorem 1.3. *If a continuous map $h : \mathbb{T}^n \to \mathbb{S}^{2k}$ is homotopic to a continuous rational map with irregularity locus of dimension at most $n - 2k - 1$, then $h^*(\sigma_{2k}) = 0$ in $H^{2k}(\mathbb{T}^n; \mathbb{Z})$.*

In Theorems 1.2 and 1.3 the assumption on the dimension of the irregularity locus cannot be relaxed, as demonstrated by the following:

Example 1.4. Let $g : \mathbb{T}^{2k} \to \mathbb{S}^{2k}$ be a smooth (of class $\mathcal{C}^\infty$) map of topological degree 1 and let $p : \mathbb{T}^n = \mathbb{T}^{2k} \times \mathbb{T}^{n-2k} \to \mathbb{T}^{2k}$ be the canonical projection. The map $h := g \circ p : \mathbb{T}^n \to \mathbb{S}^{2k}$ is homotopic to a continuous rational map with irregularity locus of dimension $n - 2k$, cf. [13, Theorem 2.4]. On the other hand, $h^*(\sigma_{2k}) \neq 0$.

2. Proofs

Let X be a nonsingular real algebraic variety. A *nonsingular complexification* of X is a nonsingular quasi-projective scheme V over $\mathbb{R}$ whose set of real points $V(\mathbb{R})$ is Zariski dense in V and biregularly isomorphic to X. The set $V(\mathbb{C})$ of complex points of V is a complex manifold of complex dimension n. The set of fixed points of complex conjugation on $V(\mathbb{C})$ is identified with $V(\mathbb{R})$. Furthermore, the set of complex points of the scheme $V_{\mathbb{C}} := V \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ is identified with $V(\mathbb{C})$. The cycle map

$$\mathrm{cl}_{\mathbb{C}} : Z^k(V_{\mathbb{C}}) \to H^{2k}\big(V(\mathbb{C}); \mathbb{Z}\big)$$

is a group homomorphism defined on the group of algebraic codimension k cycles (defined over $\mathbb{C}$) on $V_{\mathbb{C}}$, cf. [7] or [9, Chapter 19]. If $e : X \to V(\mathbb{C})$ is a map which is a biregular isomorphism between X and $V(\mathbb{R})$, the subgroup

$$H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) := e^*\big(\mathrm{cl}_{\mathbb{C}}\big(Z^k(V_{\mathbb{C}})\big)\big)$$

of $H^{2k}(X; \mathbb{Z})$ does not depend on the choice of V and e, cf. [1, p. 278]. Here

$$e^* : H^*\big(V(\mathbb{C}); \mathbb{Z}\big) \to H^*(X; \mathbb{Z})$$

is the homomorphism induced by e.

The groups $H^{2k}_{\mathbb{C}\text{-alg}}(-; \mathbb{Z})$ have the following functorial property (cf. [1, p. 278]): If $f : X \to Y$ is a regular map between nonsingular real algebraic varieties, then

$$f^*\big(H^{2k}_{\mathbb{C}\text{-alg}}(Y; \mathbb{Z})\big) \subseteq H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}).$$