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A note on metric compactifications and periodic points of maps

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ABSTRACT

In this note, we investigate metric compactifications preserving some properties of periodic points of maps. In particular, we prove that if $f : X \to X$ is any map of a locally compact and finite-dimensional separable metric space X, then there exist a metric compactification γX of X and an extension $\gamma f : \gamma X \to \gamma X$ of f such that $\dim \gamma X = \dim X$, $Cl_{\gamma X} P_i(f) = P_i(\gamma f)$ and $\dim P_i(f) = \dim P_i(\gamma f)$ for each $i \in \mathbb{N}$, where $P_i(f) = \{x \in X \mid f^i(x) = x\}$ $(= \mathrm{Fix}(f^i))$. This is the affirmative answer to Kato (2013) [9, Problem 3.7].

1. Introduction

In this note, unless stated otherwise, we assume that all spaces are separable metric spaces and all maps are continuous functions. Also, dimension means the covering dimension dim of spaces (see [5]). Let $\mathbb{N}$ be the set of all natural numbers, i.e., $\mathbb{N} = \{1, 2, \ldots\}$. For a map $f : X \to X$, let $P(f)$ be the set of all periodic points of f, i.e.,

$$P(f) = \left\{x \in X \;\middle|\; f^p(x) = x \text{ for some } p \in \mathbb{N}\right\}.$$

For each $i \in \mathbb{N}$, we put $P_i(f) = \{x \in X \mid f^i(x) = x\} (= \mathrm{Fix}(f^i))$. Note that $P(f) = \bigcup\{P_i(f) \mid i \in \mathbb{N}\}$. A subset C of X is called a *color* (see [13]) of f if $f(C) \cap C = \phi$. Note that $f(C) \cap C = \phi$ if and only if $C \cap f^{-1}(C) = \phi$. A subset C of X is called a *bright color* (see [3]) of f if $\overline{f(C)} \cap C = \phi$. Let A be any closed subset of X. We say that $f|A : A \to X$ is *colorable* (see [2]) and *brightly colorable* (see [3]) if there is a finite closed cover $\mathcal{C}$ of A such that each element C of $\mathcal{C}$ is a color and a bright color of f, respectively. A map $f : X \to X$ satisfies *the hereditarily colorable condition* provided that if $f^i|A : A \to X$ is any fixed-point free map for any closed subset A of X and any $i \in \mathbb{N}$, then $f^i|A : A \to X$ is brightly colorable. Note that for a closed map $f|A : A \to X$, $f|A$ is colorable if and only if $f|A$ is brightly colorable.

In [4], van Douwen proved that if $f : X \to X$ is a fixed-point free (i.e., $P_1(f) = \phi$) and closed map of a finite-dimensional (separable) metric space X with $\mathrm{ord}(f) = \sup\{|f^{-1}(x)| \mid x \in X\} < \infty$, then f is colorable and hence the extension $\beta f : \beta X \to \beta X$ of f is fixed-point free, where βX denotes the Stone–Čech compactification of the space X. In [9, Theorem 2.1], we proved that if $f : X \to X$ is a closed map of a finite-dimensional separable metric space X with $\mathrm{ord}(f) < \infty$, then there exist a metric compactification γX of X and an extension $\gamma f : \gamma X \to \gamma X$ of f such that $\dim \gamma X = \dim X$, $Cl_{\gamma X} P_i(f) = P_i(\gamma f)$ and $\dim P_i(f) = \dim P_i(\gamma f)$ for each $i \in \mathbb{N}$. In [2], Buzyakova and Chigogidze proved that if $f : X \to X$ is any map of a finite-dimensional, locally compact and paracompact space X, then $f : X \to X$ satisfies

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the hereditarily colorable condition (see [2, Theorem 3.4]). Also, they showed that if $f : X \to X$ is any map of a finite-dimensional, locally compact and paracompact space X, then $P_i(\beta f) = Cl_{\beta X} P_i(f)$. In [2, Proposition 4.4], they showed that there exists a non-colorable fixed-point free map $g : U \to X$ such that U is an open subset of a (zero-dimensional) compact metric space X. The properties of colors of maps have been studied by many authors (see [1–4,6–13]).

First, we prove the following theorem whose proof is a modification of the proof of [9, Theorem 2.1]. We need more precise argument.

Theorem 1.1. *If $f : X \to X$ is a map of a separable metric space X and f satisfies the hereditarily colorable condition, then there exist a metric compactification γX of X and an extension $\gamma f : \gamma X \to \gamma X$ of f such that $\dim \gamma X = \dim X$, $Cl_{\gamma X} P_i(f) = P_i(\gamma f)$ and $\dim P_i(f) = \dim P_i(\gamma f)$ for each $i \in \mathbb{N}$.*

Proof. We put $\mathcal{P} = \{P_i(f) \mid i \in \mathbb{N}\}$ and enumerate $\mathcal{P}$ as $\{Q_j \mid j \in \mathbb{N}\}$ such that if $P \in \mathcal{P}$, then $|\{j \in \mathbb{N} \mid P = Q_j\}| = \infty$. i.e., each $P \in \mathcal{P}$ is listed infinitely often in $\{Q_j \mid j \in \mathbb{N}\}$. Then we have a Wallman compactification $\alpha_1 X = \omega(X, \mathcal{A}_1)$ such that $\mathcal{A}_1$ is a countable Wallman (closed) base for X and $\dim \alpha_1 X = \dim X$ and $\dim Cl_{\alpha_1 X}(Q_1) = \dim Q_1$, i.e.,

$$\omega(X, \mathcal{A}_1) = \{\mathcal{S} \subset \mathcal{A}_1 \mid \mathcal{S} \text{ is an } \mathcal{A}_1\text{-ultrafilter}\}$$

(see [13, Theorem 3.5.3]). Also, we may assume that $P_i(f) \in \mathcal{A}_1$ for all $i \in \mathbb{N}$. For each $F \in \mathcal{A}_1$, let

$$F^\star = \{\mathcal{S} \in \omega(X, \mathcal{A}_1) \mid F \in \mathcal{S}\}.$$

Then we know that $F^\star = Cl_{\alpha_1 X}(F)$. We take a Wallman compactification $s_1 X = \omega(X, \mathcal{B}_1)$ of X and an extension $s_1 f : s_1 X \to s_1 X$ of f such that $\mathcal{B}_1$ is a countable Wallman base for X and $\mathcal{A}_1 \subset \mathcal{B}_1$. Suppose $Q_1 = P_{i_1}(f)$ for some $i_1 \in \mathbb{N}$. Since $\mathcal{B}_1^\star = \{F^\star \mid F \in \mathcal{B}_1\}$ is a countable closed base of the compact metric space $s_1 X = \omega(X, \mathcal{B}_1)$, we can find a sequence $\{F_j \mid j \in \mathbb{N}\}$ of $\mathcal{B}_1$ such that $F_j^\star \subset \mathrm{Int}_{s_1 X} F_{j+1}^\star$ and

$$\bigcup_{j \in \mathbb{N}} F_j^\star = \omega(X, \mathcal{B}_1) - Q_1^\star.$$

Note that for each $j \in \mathbb{N}$, $f^{i_1}|F_j$ is a fixed-point free map. Since f satisfies the hereditarily colorable condition, $f^{i_1}|F_j : F_j \to X$ is brightly colorable. Hence there is a finite closed cover $\mathcal{H}_j$ of F_j such that $Cl_X f^{i_1}(H) \cap H = \phi$ for each $H \in \mathcal{H}_j$. Put

$$\mathcal{K}_1 = \{H \mid H \in \mathcal{H}_j,\ j \in \mathbb{N}\} \cup \{Cl_X f^{i_1}(H) \mid H \in \mathcal{H}_j,\ j \in \mathbb{N}\}.$$

Note that $\mathcal{K}_1$ is countable. Then we can choose a countable Wallman base $\mathcal{C}_1$ such that $\mathcal{K}_1 \cup \mathcal{B}_1 \subset \mathcal{C}_1$ and $f^{-1}(\mathcal{C}_1) \subset \mathcal{C}_1$. Put $t_1 X = \omega(X, \mathcal{C}_1)$. Then we have the unique extension $t_1 f : t_1 X \to t_1 X$ of f. Let $a_1 : s_1 X \to \alpha_1 X$ and $b_1 : t_1 X \to s_1 X$ be the extensions of $id_X : X \to X$ respectively. Note that $s_1 f \cdot b_1 = b_1 \cdot t_1 f$.

We shall show that

$$b_1(P_{i_1}(t_1 f)) = Cl_{s_1 X} P_{i_1}(f) (= Q_1^\star).$$

Note that $b_1(Cl_{t_1 X} P_{i_1}(f)) = Cl_{s_1 X} P_{i_1}(f)$. Let $q \in P_{i_1}(t_1 f)$. Suppose, on the contrary, that $b_1(q) \in s_1 X - Cl_{s_1 X}(P_{i_1}(f))$. We have some F_j such that $b_1(q) \in F_j^\star$. Then we can take $G_{j+1} \in \mathcal{B}_1$ such that

$$G_{j+1}^\star \cup F_{j+1}^\star = \omega(X, \mathcal{B}_1) (= s_1 X), \qquad G_{j+1}^\star \cap F_j^\star = \phi.$$

Then $F_{j+1} \cup G_{j+1} = X$. Hence $F_{j+1}^\star \cup G_{j+1}^\star = t_1 X$, $b_1(F_{j+1}^\star) \subset F_{j+1}^\star$ and $b_1(G_{j+1}^\star) \subset G_{j+1}^\star$. Since $b_1(q) \notin G_{j+1}^\star$, we see that $q \in F_{j+1}^\star (\subset t_1 X)$. Since $F_{j+1}^\star = \bigcup\{H^\star \mid H \in \mathcal{H}_{j+1}\}$, there is $H \in \mathcal{H}_{j+1}$ such that $q \in H^\star$. Note that

$$(t_1 f)^{i_1}(q) \in (t_1 f)^{i_1}(H^\star) \subset \overline{(t_1 f)^{i_1}(H)} = \overline{f^{i_1}(H)} \subset \overline{Cl_X f^{i_1}(H)} = [Cl_X f^{i_1}(H)]^\star.$$

Since $Cl_X f^{i_1}(H) \cap H = \phi$ and $Cl_X f^{i_1}(H), H \in \mathcal{C}_1$, we see that $[Cl_X f^{i_1}(H)]^\star \cap H^\star = \phi$ and hence $(t_1 f)^{i_1}(q) \neq q$. This implies that $q \notin P_{i_1}(t_1 f)$, which is a contradiction.

Also, we can choose a Wallman compactification $\alpha_2 X = \omega(X, \mathcal{A}_2)$ such that

(1) $\mathcal{A}_2$ is a countable Wallman (closed) base for X,
(2) $\dim \alpha_2 X = \dim X$, $\dim Cl_{\alpha_2 X}(Q_2) = \dim Q_2$, and
(3) $\mathcal{C}_1 \subset \mathcal{A}_2$ (see [13, Theorem 3.5.3]).

Then we have the extension $c_1 : \alpha_2 X \to t_1 X$ of $id_X : X \to X$. This is the first step.