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Filtrations induced by continuous functions

B. Di Fabio *, P. Frosini

ARCES and Dipartimento di Matematica, Università di Bologna, Italy

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A B S T R A C T

In Persistent Homology and Topology, filtrations are usually given by introducing an ordered collection of sets or a continuous function from a topological space to $\mathbb{R}^n$. A natural question arises, whether these approaches are equivalent or not. In this paper we study this problem and prove that, while the answer to the previous question is negative in the general case, the approach by continuous functions is not restrictive with respect to the other, provided that some natural stability and completeness assumptions are made. In particular, we show that every compact and stable 1-dimensional filtration of a compact metric space is induced by a continuous function. Moreover, we extend the previous result to the case of multi-dimensional filtrations, requiring that our filtration is also complete. Three examples show that we cannot drop the assumptions about stability and completeness. Consequences of our results on the definition of a distance between filtrations are finally discussed.

0. Introduction

The concept of filtration is the start point for Persistent Topology and Homology. Actually, the main goal of these theories is to examine the topological and homological changes that happen when we go through a family of spaces that is totally ordered with respect to inclusion [12]. In literature, filtrations are usually given in two ways. The former consists of explicitly introducing a nested collection of sets (usually carriers of simplicial complexes), the latter of giving a continuous function from a topological space to $\mathbb{R}$ or $\mathbb{R}^n$ (called a *filtering function*), whose sub-level sets represent the elements of the considered filtration (cf., e.g., [11,15]). An example of these two types of filtrations is shown in Fig. 1. The two considered methods have produced two different approaches to study the concept of persistence. A natural question arises, whether these approaches are equivalent or not. In our paper we study this problem and prove that, while the answer to the previous question is negative in the general case, the approach by continuous functions is not restrictive with respect to the other, provided that some natural stability and completeness assumptions are made. In some sense, this statement shows that the approach by continuous functions (and the related theoretical properties) can be used without loss of generality, and represents the main result of this paper.

The interest in this investigation is mainly due to the desire of building a bridge between the two settings, which would ensure that results available in literature for the approach by functions are also valid for the other method. As examples of results that have been proved in one setting and that it would be desirable to apply to the other, we can cite [5] and [4], in which persistence diagrams in the 1-dimensional and n-dimensional setting, respectively, are proved to be stable shape descriptors via the use of the associated filtering functions. Another example can be found in [6], where a Mayer–Vietoris formula involving the ranks of persistent homology groups of a space and its subspaces is obtained by defining a filtering function for the union space and taking account of its restrictions to the considered subspaces.

* Corresponding author.
E-mail addresses: barbara.difabio@unibo.it (B. Di Fabio), patrizio.frosini@unibo.it (P. Frosini).

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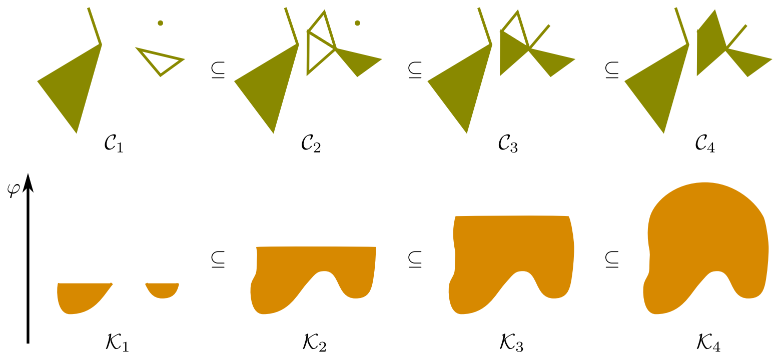

Fig. 1. Examples of filtrations. First row: nested carriers of simplicial complexes $\{\mathcal{C}_i\}$. Second row: the sub-level sets $\{\mathcal{K}_i\}$ of a real-valued continuous function φ.

Another important reason which drives our investigation is related to the problem of defining a distance between different filtrations of the same space. Nowadays, this problem is usually tackled by translating the direct comparison between two filtrations into the comparison of the associated persistence diagrams through the study of persistent homology. Unfortunately, there exist some simple examples showing that this kind of comparison is not always able to distinguish two different filtrations (see e.g. [1,13,7]). For this reason, our idea is to define a distance between filtrations in terms of a distance between the associated filtering functions, and to this scope, we need to prove that each filtration is induced by at least one function (see Section 4).

In this paper we just consider stable filtrations. The property of stability of a filtration we ask for is motivated by the fact that in real applications we need to work with methodologies that are robust in the presence of noise. As a consequence, we have to require that the inclusions considered in our filtration persist under the action of small perturbations. For the same reason, we also need that a small change of the parameter in our filtration (whenever applicable) does not produce a large change of the associated set with respect to the Hausdorff distance. These assumptions are formalized by our definition of stable filtration (Definition 2.1).

In order to make our treatment as general as possible, we just require that the sets $\mathcal{K}_i$ $(i \in I)$ in our filtration are compact subsets of a compact metric space K, and that the indexing set I is compact.

The paper starts by considering filtrations indexed by a 1-dimensional parameter. In this setting, after proving some lemmas, we show that every compact and stable 1-dimensional filtration of a compact metric space is induced by a continuous function (Theorem 2.8). In the last part of the paper, this result is extended to the case of multi-dimensional filtrations (Theorem 3.4), i.e. the case of filtrations indexed by an n-dimensional parameter (cf. [2,3]). In order to do that, we need to assume also that our filtration is complete, i.e. compatible with respect to intersection (Definition 3.2). Three examples show that we cannot drop either the assumption about stability or the one concerning completeness (Examples 1, 2 and 3). Some considerations on the consequences of our results conclude the paper.

1. Preliminaries

In this section we give the preliminary concepts and the notation that will be used throughout the paper.

Let (K, d) be a compact metric space containing at least two points. Let us denote by $Comp(K)$ the set $\{\mathcal{K}\colon \mathcal{K}$ compact in $K\}$. Let us consider the Hausdorff distance d_H on $Comp(K) \setminus \{\emptyset\}$, and extend d_H to a distance on $Comp(K)$ by setting $d_H(\emptyset, \emptyset) = 0$ and $d_H(\emptyset, \mathcal{K}) = d_H(\mathcal{K}, \emptyset) = \mathrm{diam}(K)/2$ for every $\mathcal{K} \in Comp(K) \setminus \{\emptyset\}$. In what follows, we will still denote this distance by d_H, and call it the Hausdorff distance on $Comp(K)$. Moreover, let I be a non-empty subset of $\mathbb{R}^n$ such that $I = I_1 \times I_2 \times \cdots \times I_n$. The following relation $\preccurlyeq$ is defined in $\mathbb{R}^n$: for $i = (i_1, \ldots, i_n)$, $i' = (i'_1, \ldots, i'_n) \in \mathbb{R}^n$, we say $i \preccurlyeq i'$ if and only if $i_r \leqslant i'_r$ for every $r = 1, \ldots, n$.

Definition 1.1. An n-dimensional *filtration* of K is an indexed family $\{\mathcal{K}_i \in Comp(K)\}_{i \in I}$ such that, $\emptyset, K \in \{\mathcal{K}_i\}_{i \in I}$, and $\mathcal{K}_i \subseteq \mathcal{K}_{i'}$ for every $i, i' \in I$, with $i \preccurlyeq i'$.

Definition 1.2. An n-dimensional filtration $\{\mathcal{K}_i\}_{i \in I}$ of K is *induced by a function* $\vec{\varphi} : K \to \mathbb{R}^n$ if $\mathcal{K}_i = \{P \in K\colon \vec{\varphi}(P) \preccurlyeq i\}$ for every $i \in I$.

Definition 1.3. We shall call *compact*, or *finite* any filtration $\{\mathcal{K}_i\}_{i \in I}$ with $I = I_1 \times I_2 \times \cdots \times I_n$ a compact, or finite subset of $\mathbb{R}^n$, respectively.

Remark 1.4. When I is bounded, the assumption that $\emptyset, K \in \{\mathcal{K}_i\}_{i \in I}$ is not so restrictive, since each family of compact sets verifying the last property in Definition 1.1 can be extended to a family containing $\emptyset$ and K, without losing that property. This assumption allows us a more concise exposition.