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A non-commutative Priestley duality

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ABSTRACT

We prove that the category of left-handed strongly distributive skew lattices with zero and proper homomorphisms is dually equivalent to a category of sheaves over local Priestley spaces. Our result thus provides a non-commutative version of classical Priestley duality for distributive lattices and generalizes the recent development of Stone duality for skew Boolean algebras.

From the point of view of skew lattices, Leech showed early on that any strongly distributive skew lattice can be embedded in the skew lattice of partial functions on some set with the operations being given by restriction and so-called override. Our duality shows that there is a canonical choice for this embedding.

Conversely, from the point of view of sheaves over Boolean spaces, our results show that skew lattices correspond to Priestley orders on these spaces and that skew lattice structures are naturally appropriate in any setting involving sheaves over Priestley spaces.

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1. Introduction

Skew lattices [19,20] are a non-commutative version of lattices: algebraically, a skew lattice is a structure $(S, \vee, \wedge)$, where $\vee$ and $\wedge$ are binary operations which satisfy the associative and idempotent laws, and certain absorption laws (see Section 2.1 below).

Concrete classes of examples of skew lattices occur in many situations. The skew lattices in such classes of examples often have a *zero* element, and also satisfy certain additional axioms, which are called *strong distributivity* and *left-handedness* (see Sections 2.3 and 2.4 below). A (proto)typical class of such examples is that of *skew lattices of partial functions*, which we will describe now. If X and Y are sets, then the collection S of partial functions from X to Y carries a natural skew lattice structure, as follows. If $f, g \in S$ are partial functions, we define $f \wedge g$ to be the *restriction* of f by g, that is, the function with domain $\mathrm{dom}(f) \cap \mathrm{dom}(g)$, where its value is defined to be equal to the value of f. We define $f \vee g$ to be the *override* of f with g, that is, the function with domain $\mathrm{dom}(f) \cup \mathrm{dom}(g)$, where its value is defined to be equal to the value of g whenever g is defined, and to the value of f otherwise. The *zero element* is the unique function with empty domain.

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One consequence of the results in this paper is that *every left-handed strongly distributive skew lattice with zero can be embedded into a skew lattice of partial functions*. This fact was first proved in [21, 3.7] as a consequence of the description of the subdirectly irreducible algebras in the variety of strongly distributive skew lattices. Our proof will not depend on this result, and it will moreover provide a canonical choice of an enveloping skew lattice of partial functions. A related result in computer science is described in [2], where the authors give a complete axiomatization of the structure of partial functions with the operations override and 'update', from which the 'restriction' given above can also be defined.

In order to state our results precisely, background is needed in skew lattices, Priestley duality, and sheaf theory (see Sections 2 and 3). In particular, we make essential use of the well-known correspondence between étalé spaces and sheaves. This correspondence allows one to view a sheaf over a space X as a bundle $p : E \to X$ of sets $\{p^{-1}(x)\}_{x\in X}$ such that p is a local homeomorphism (see Section 3.4 below). The *local sections* of the sheaf are then the partial maps from X to E for which the image of each x in the domain belongs to the stalk $p^{-1}(x)$. This is how sheaves give rise to partial maps. The set of all local sections with clopen domains forms a skew Boolean algebra and if X is also equipped with a partial order, then the local sections with domains that are clopen downsets form a strongly distributive skew lattice. Our duality shows that this accounts for all strongly distributive skew lattices: we will prove that *every left-handed strongly distributive skew lattice with zero is isomorphic to a skew lattice of all local sections over clopen downsets of some bundle*. Moreover, it will be a consequence of our duality result that there is a canonical choice for the bundle and base space which represent a given skew lattice. Among all representing bundles, there is an (up to isomorphism) unique bundle $p : E \twoheadrightarrow X$ such that p is a local homeomorphism (i.e., *étale* map) and X is a *local Priestley space* (a space whose one-point-compactification is a Priestley space, see Section 3.3 below). This result generalizes both Priestley duality [26] and recent results on Stone duality [27] for skew Boolean algebras [1,13,15], see also [14].

Thus, in any setting where sheaves over Priestley spaces are present, in addition to whatever other structure, strongly distributive skew lattice structures are intrinsic. Let us name two examples of settings where sheaves over Priestley spaces are (implicitly) present. First, the classical representation of commutative unital rings as sheaves over their prime ideal spectra with the Zariski topology: Hochster [11] showed that the topological spaces which arise as prime ideal spectra are *exactly* the spaces which arise as the Stone duals of distributive lattices, which are now known as *spectral* spaces. Much more recently [9], a sheaf representation over spectral spaces was obtained for MV-algebras, whose category is equivalent to a subcategory of lattice-ordered abelian groups. To place these results precisely in the setting of this paper, it suffices to remark that the category of spectral spaces and spectral functions is *isomorphic* to the category of Priestley spaces and continuous monotone functions (cf. [6,3]). Therefore, any sheaf representation over a spectral space can be equivalently regarded as a sheaf representation over a Priestley space.

In conclusion, our results show that the embeddability of strongly distributive skew lattices in partial function algebras is not coincidental, but a fully structural and natural phenomenon. They also show that strongly distributive skew lattices are intrinsic to sheaves over Priestley spaces and that each such lattice has a canonical embedding into a skew Boolean algebra, namely the skew Boolean algebra of all local sections with clopen domains over the corresponding base. Thus our results open the way to exploring the logic of such structures. In particular, they provide a candidate notion of Booleanization which may in turn allow the development of a non-commutative version of Heyting algebras.

The paper is organized as follows. We first provide background on skew lattices (Section 2), Priestley duality (Section 3), and sheaves (Section 3.4). After these preliminaries, we will be ready to state our main theorem (Theorem 3.7), that the categories of left-handed strongly distributive skew lattices and sheaves over local Priestley spaces are dually equivalent. Starting the proof of this theorem, we first give a more formal description of the skew lattice of local sections of an étalé space, and show that it gives rise to a functor (Section 4). To show that this functor is part of a dual equivalence, we will describe how to reconstruct the étalé space from its skew lattice of local sections (Section 5), and give a general description of this process for an arbitrary left-handed strongly distributive skew lattice (Section 6). Finally (Section 7), we will put together the results from the preceding sections to prove our main theorem. We close with a few concluding remarks (Section 8).

2. The category SDL of strongly distributive left-handed skew lattices

For an extensive introduction to the theory of skew lattices we refer the reader to [19–22]. To make our exposition self-contained, we collect some definitions and basic facts of the theory.

2.1. Skew lattices

A *skew lattice*[1] S is an algebra $(S, \wedge, \vee, 0)$ of type $(2, 2, 0)$, such that the operations $\wedge$ and $\vee$ are associative, idempotent and satisfy the absorption identities

$$x \wedge (x \vee y) = x = x \vee (x \wedge y),$$

$$(y \vee x) \wedge x = x = (y \wedge x) \vee x,$$

and the 0 element satisfies $x \wedge 0 = 0 = 0 \wedge x$. Note that a *lattice* is a skew lattice in which $\wedge$ and $\vee$ are commutative.

1 In this paper, all skew lattices will be assumed to have a zero element.

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